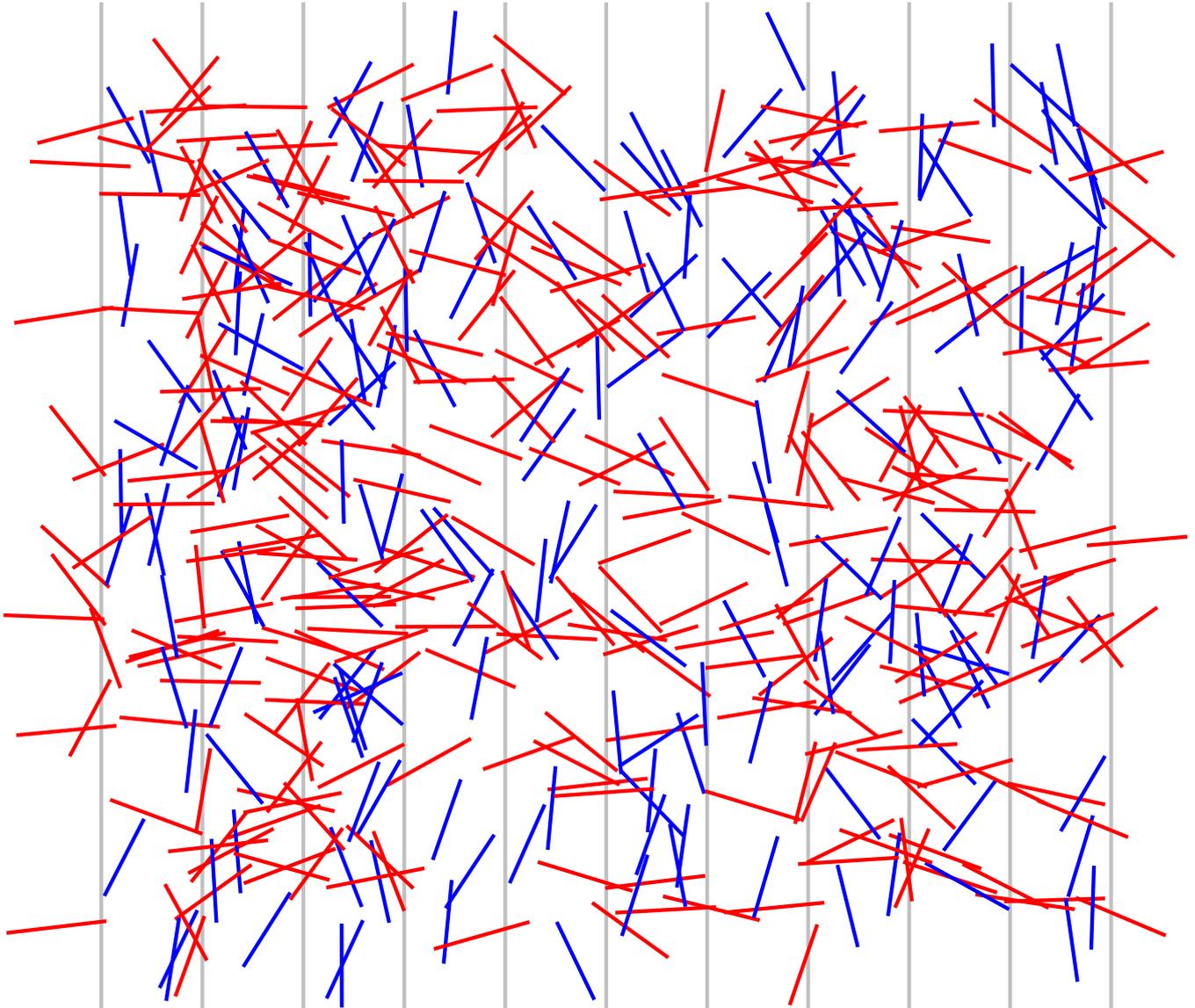


Buffon's Needle

A Collection of Mathematical Investigations



by Kevin Bertman

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Introduction

In the 18th century, Georges-Louis Leclerc, Comte de Buffon, investigated the probability of dropping a needle onto equally spaced strips so that the needle crossed the boundary between two strips.

Consider figure 1. The red needle has crossed the boundary between two strips while the blue one hasn't.

When the length of the needle is equal to the spacing between the strips Leclerc was able to show that the probability that a needle would cross a boundary is equal to $\frac{2}{\pi}$.



Georges-Louis Leclerc,
Comte de Buffon

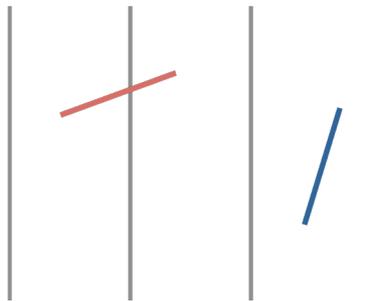


Figure 1

This document contains four investigations based on Buffon's Needle. Each investigation is designed as a stand-alone document that can be given to students (hence a lot of the information is repeated, and figure labels are reset in each section).

The first investigation asks students to perform an experiment and use simple probability to estimate the value of π .

The second investigation asks students to design, create and perform a virtual experiment using Excel and use the results to estimate the value of π .

The third investigation asks students to set up a model in Geogebra, and then use this model to approximate the value of π .

The fourth investigation asks students to use the definite integral to show that the probability that a needle will cross a boundary is equal to $\frac{2}{\pi}$.

Teacher notes for each investigation are included, along with Excel and Geogebra tutorials, and a worksheet that derives the limit form of the definite integral.

Estimating the Value of Pi Using Experimental Probability

In the 18th century, Georges-Louis Leclerc, Comte de Buffon, investigated the probability of dropping a needle onto equally spaced strips so that the needle crossed the boundary between two strips.



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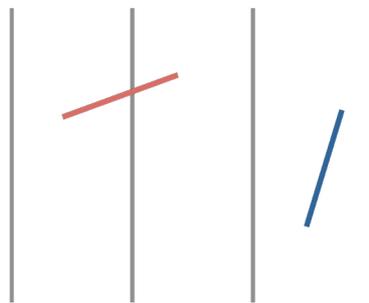


Figure 1

1. Get a small cocktail stick and a large sheet of paper. Using a ruler and pencil, divide the paper into equally spaced strips. The width of each strip should be equal to the length of the stick.
2. Drop the “needle” from a height of 1 metre onto the paper. Record whether or not it crosses the boundary between two strips. Repeat this 200 times. Copy and complete the following table as you go. The first two rows have been filled in *as an example*. Your results may be different.

Number of Needles Dropped	Crossed Boundary Total	Experimental Probability of Crossing Boundary
20	13	$\frac{13}{20} = 0.65$
40	25	$\frac{25}{40} = 0.625$
60		
80		
100		
120		
140		
160		
180		
200		

3. Answer the following questions:

- Did your needle always land on the sheet of paper? What did you do if it didn't?
- Remember that Leclerc was able to show the probability that a needle crossed a boundary was equal to $\frac{2}{\pi}$. Use this fact to estimate the value of π at each stage of your experiment (after dropping 20 needles, after dropping 40 needles etc.) Show your working out.
- Compare your estimated values of π with the actual value of π to 2 decimal places. Are your results accurate? Do you notice any patterns?
- Collect everyone in the class's crossed boundary total from their last row. Use these results to estimate the value of π . Is it more accurate than your results in part c)? If you would like to include even more results you may use figures 2 to 5. Each figure shows the results of the experiment performed 200 times.

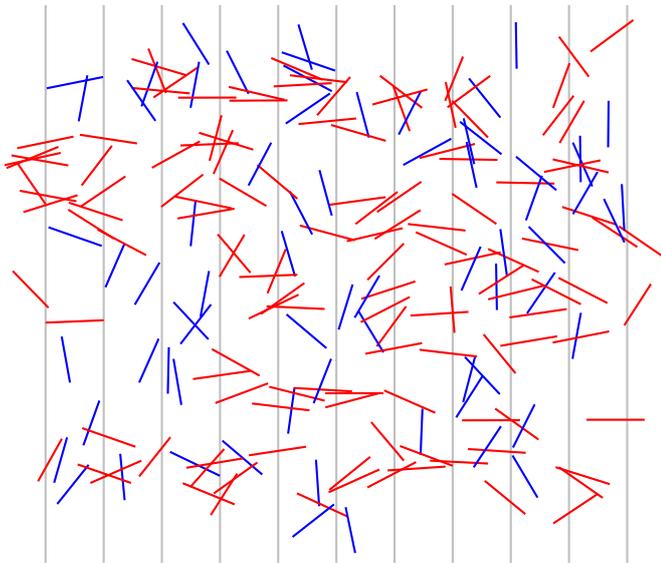


Figure 2

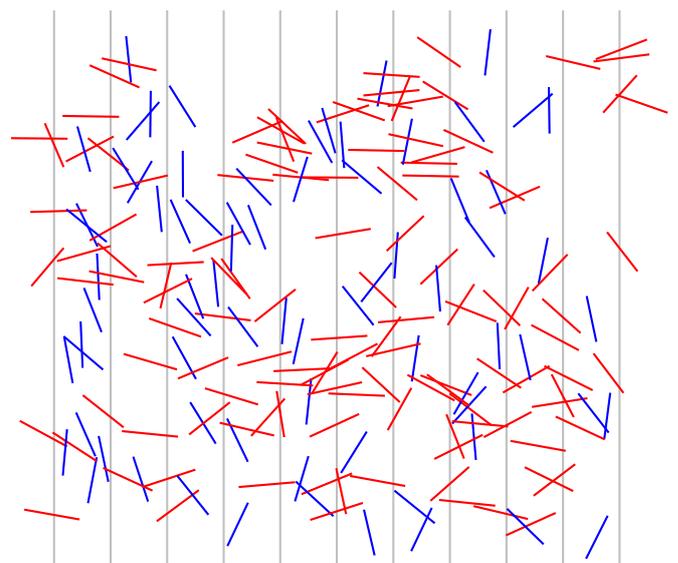


Figure 3

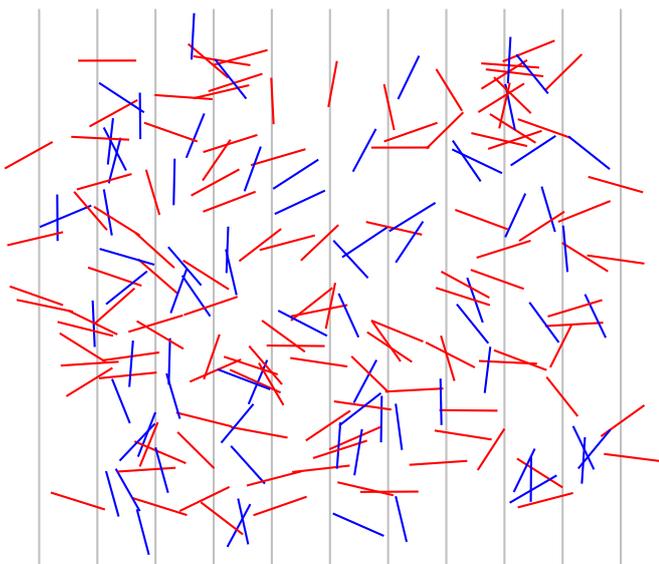


Figure 4

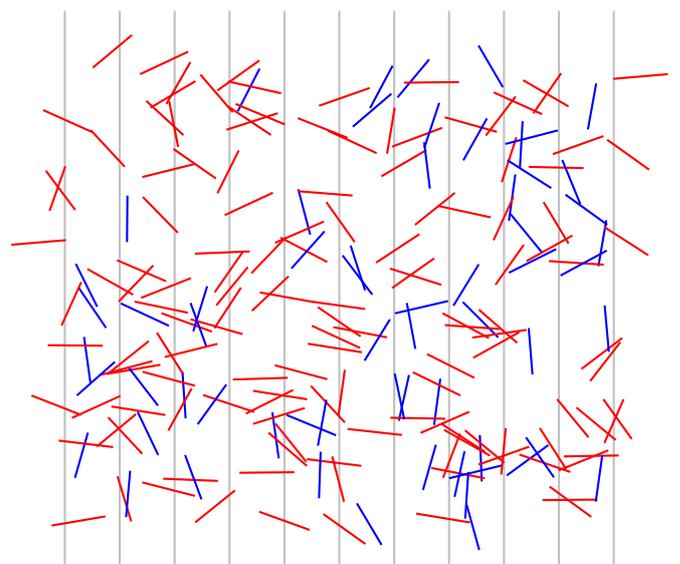


Figure 5

Creating a Virtual Experiment to Estimate the Value of Pi

In the 18th century, Georges-Louis Leclerc, Comte de Buffon, investigated the probability of dropping a needle onto equally spaced strips so that the needle crossed the boundary between two strips.

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Comte de Buffon

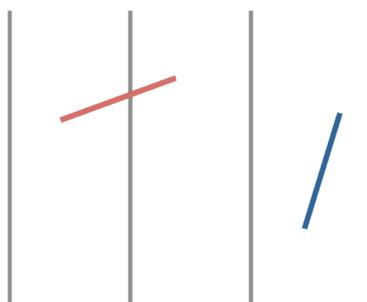


Figure 1

Figure 2 shows the results of the experiment performed 1000 times.

Use Excel to simulate the experiment thousands of times. Your spreadsheet can be entirely numerical, or a combination of numerical and visual (as in figure 2). Use your results to estimate the value of π .

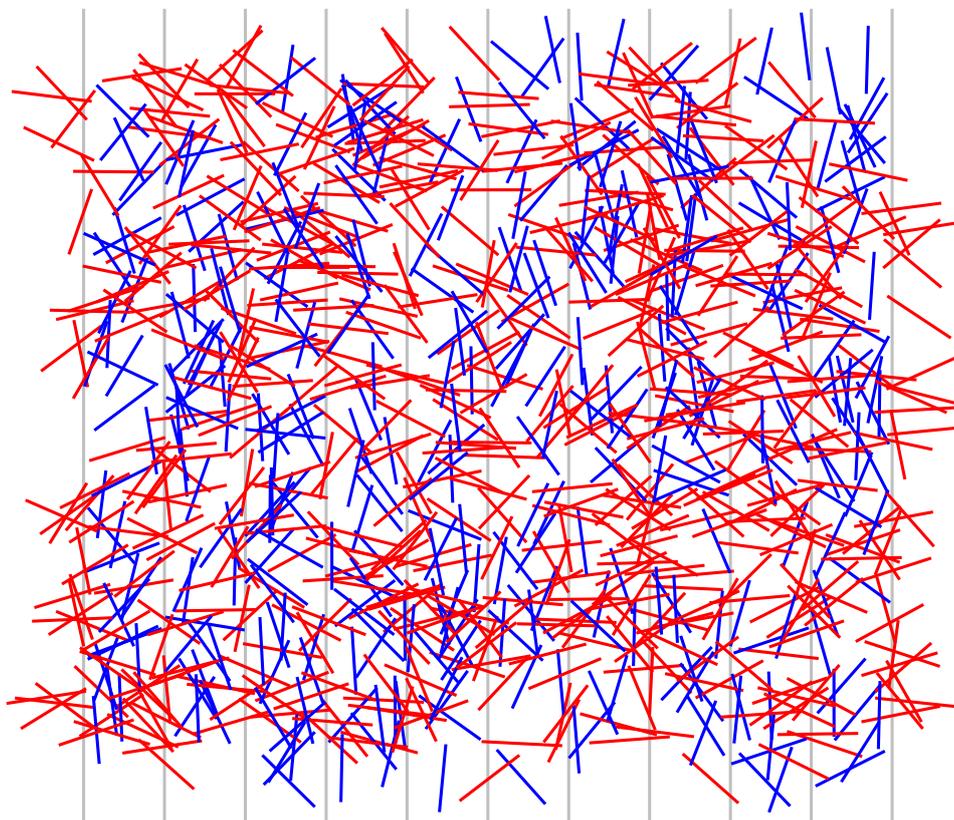


Figure 2

Approximating Pi Using Geogebra

In the 18th century, Georges-Louis Leclerc, Comte de Buffon, investigated the probability of dropping a needle onto equally spaced strips so that the needle crossed the boundary between two strips.

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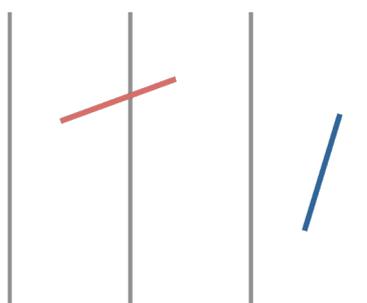


Figure 1

The y -coordinate does not affect whether a needle crosses the boundary, so we can limit the needle to landing on a horizontal line. Let us start by limiting the number of points where the needle can land. Figure 2 uses nine equally spaced points. The shaded area shows where the needle will have to lay in order to cross a boundary if it lands on the third point.

The probability that the needle will land on this point and lay within the shaded area will be equal to

$$\frac{1}{9} \times \frac{\angle APB + \angle DPC}{360}$$

If we perform this calculation for each point we can estimate the probability of the needle crossing a boundary. The more points on which we allow the needle to land, the better our estimation will be.

Using Geogebra:

- construct a diagram like figure 2. As point P moves horizontally the shaded area should adjust automatically to show where the needle must land in order to cross a boundary.
- animate point P so it automatically moves from the left of the horizontal line to the right. During each frame of the animation Geogebra should automatically record angles $\angle APB$ and $\angle DPC$ to the spreadsheet. Use these recorded values to estimate the value of π

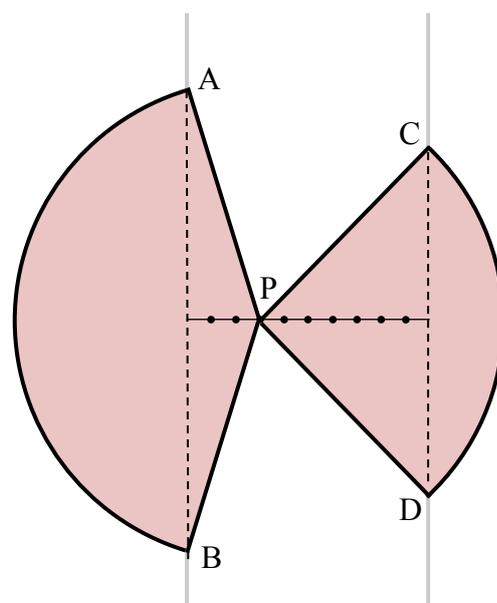


Figure 2

Buffon's Needle and the Definite Integral

In the 18th century, Georges-Louis Leclerc, Comte de Buffon, investigated the probability of dropping a needle onto equally spaced strips so that the needle crossed the boundary between two strips.



Georges-Louis Leclerc,
Comte de Buffon

Consider figure 1. The red needle has crossed the boundary between two strips while the blue one hasn't.

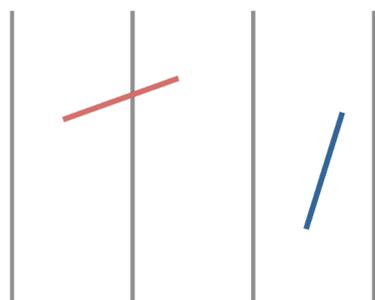


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$$\frac{1}{9} \times \frac{\angle APB + \angle DPC}{360}$$

If we perform this calculation for each point we can estimate the probability of the needle crossing a boundary. The more points on which we allow the needle to land, the better our estimation will be.

Use this method to calculate the exact value that a needle will cross a boundary. Recall the limit definition of the definite integral. You will need to use this:

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x) \cdot \Delta x$$

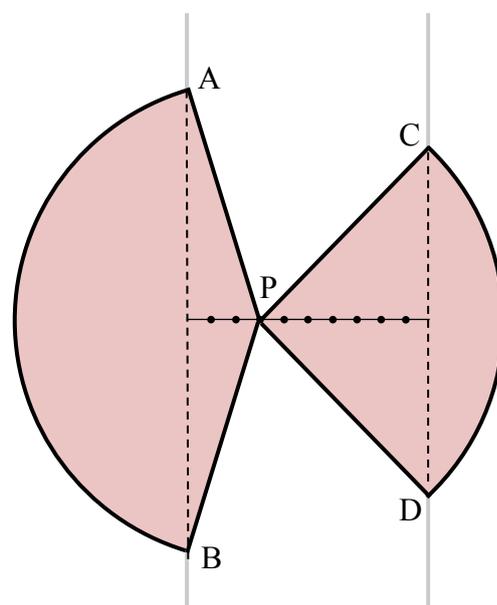


Figure 2

Appendix A: Excel Tutorial

Relative References

Type the following values into cells A1 to C4:

	A	B	C	D
1	3	5	2	
2	1	8	6	
3	2	3	9	
4	5	6	1	

Type the following equation into cell D1:

	A	B	C	D
1	3	5	2	=A1+3
2	1	8	6	
3	2	3	9	
4	5	6	1	

Press *Enter* and the value displayed in cell D1 should be equal to 6.

Copy and paste cell D1 to cell D2:

	A	B	C	D	E
1	3	5	2	6	
2	1	8	6	4	
3	2	3	9		
4	5	6	1		

Notice that the equation in cell D2 becomes A2+3. This is because we pasted **one** row below cell D1, so the reference A1 becomes A2 (the row number increases by **one**).

Copy and paste cell D1 into cell F4:

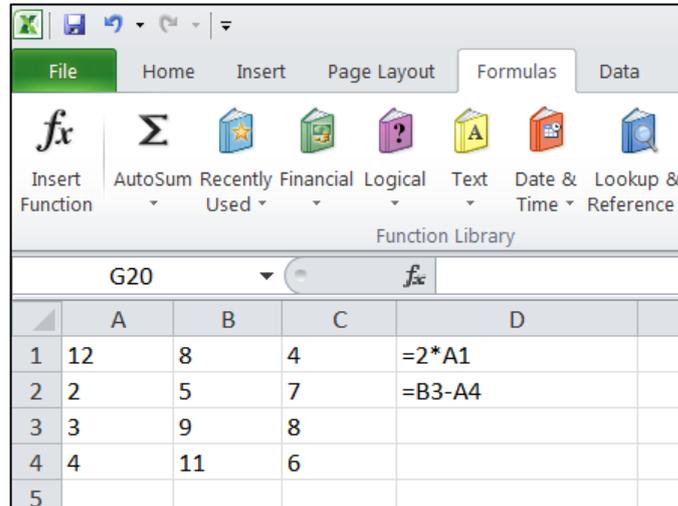
	A	B	C	D	E	F	G
1	3	5	2	6			
2	1	8	6	4			
3	2	3	9				
4	5	6	1			4	

Notice that the equation in cell F4 becomes C4+3. This is because we pasted **two** columns to the right and **three** rows below cell D1, so the reference A1 becomes C4 (the column letter increases by **two** and the row number increases by **three**).

These are examples of *relative references*.

Task 1

The following screenshot shows the contents of various cells (Note that you will not see the equations on your screen after you press *Enter*).



Complete the following table *without* using Excel. Then, check your answers using Excel.

For example, if we copy and paste cell D1 to cell E2, cell E2 will contain the equation $2*B2$ and will display the value of 10.

Copied From	Pasted To	Displayed Equation	Displayed Value
D1	E2	$2*B2$	10
D1	D4		
D1	E3		
D2	E1		
D2	F2		
D2	E3		

Absolute References

Type the following values in cells A1 to C4:

	A	B	C
1	3	8	9
2	2	6	4
3	1	7	2
4	5	3	8

Type the following equations into cells D1 to D3:

	A	B	C	D
1	3	8	9	=A\$1
2	2	6	4	=A1
3	1	7	2	=A\$1
4	5	3	8	
5				

Copy and paste cell D1 to E3:

	A	B	C	D	E	F
1	3	8	9	3		
2	2	6	4	3		
3	1	7	2	3	3	
4	5	3	8			
5						

Notice that even though we pasted to a new column and row, the cell reference doesn't change.

Placing a \$ sign in front of the column letter means the column letter will never change.

Placing a \$ sign in front of the row number means the row number will never change.

Copy and paste cell D2 to E4:

	A	B	C	D	E	F
1	3	8	9	3		
2	2	6	4	3		
3	1	7	2	3	3	
4	5	3	8		1	
5						
6						

Since there is a \$ sign in front of the column letter, the column letter will never change.

Cell E4 is **two** rows below cell D2, so the row number increases by **two**.

Copy and paste cell D3 into cell F4:

	A	B	C	D	E	F	G
1	3	8	9	3			
2	2	6	4	3			
3	1	7	2	3	3		
4	5	3	8		1	9	
5							
6							

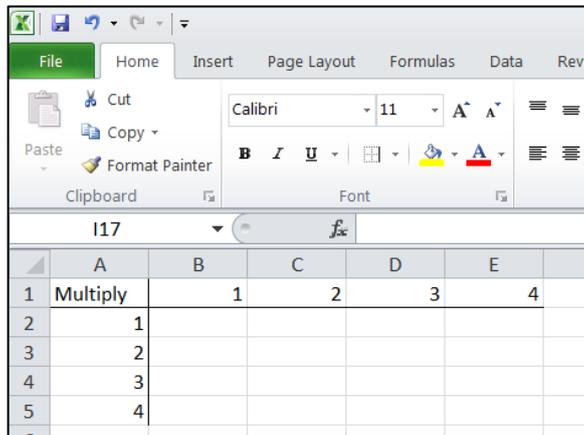
Since there is a \$ sign in front of the row number, the row number will never change.

Cell F4 is **two** columns to the right of cell D3, so the column letter increases by **two**.

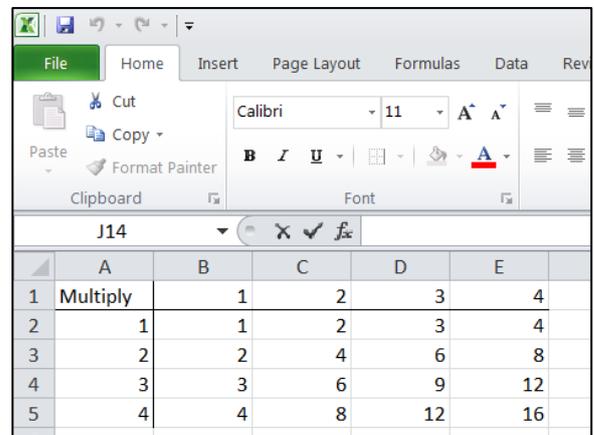
These are examples of *absolute references*.

Task 2

Enter one equation into cell B2, so that when it is copied and pasted into all of the other cells it correctly fills in the multiplication table:



The screenshot shows an Excel spreadsheet with a multiplication table. The table has columns labeled A through E and rows labeled 1 through 5. Cell A1 contains the text "Multiply". The other cells in the table contain the products of the row and column numbers. For example, cell B2 contains the value 1, C2 contains 2, D2 contains 3, and E2 contains 4. The formula bar shows the formula in cell B2 is $=A2*B1$.



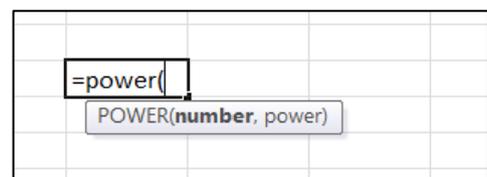
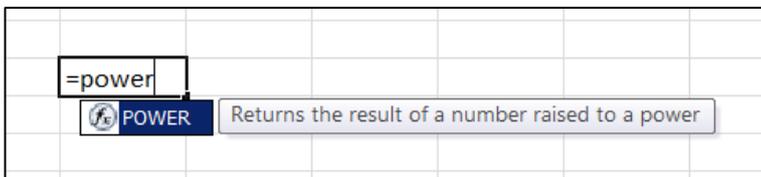
The screenshot shows the same Excel spreadsheet after the formula from cell B2 has been copied and pasted into all other cells. The multiplication table is now complete. For example, cell B3 contains 2, C3 contains 4, D3 contains 6, and E3 contains 8. The formula bar shows the formula in cell B3 is $=A3*B1$.

Mathematical Functions

Excel contains many mathematical functions. For example:

$\sin()$, $\cos()$, $\tan()$, $\text{asin}()$, $\text{acos}()$, $\text{atan}()$, $\text{sqrt}()$, $\text{power}()$, $\text{int}()$, $\text{randbetween}()$, $\text{sum}()$, $\text{average}()$, $\text{countif}()$

As you type the function into Excel, you will often be given a description of the function and what kind of values it accepts:



Examples

- Typing $=\text{power}(3,4)$ will display the value of 81
- Type $=\text{sqrt}(9)$ will display the value of 3
- Typing $=\text{sum}(A1:A6)$ will calculate the sum of the contents of cells A1 to A6

Task 3

Use Excel to determine the following:

- a) the square root of 20
- b) the mean value of 20, 18, 23, 11, 25 and 19
- c) a randomly generated number between 5 and 13
- d) the inverse sine of 1
- e) the sum of all positive integers less than 10

Conditional Logic

Excel contains an IF statement that works according to the following:

`=IF(condition, value to display if true, value to display if false)`

For example, in cell A1 type `=IF(A2<A3,"Cell A2 is less than cell A3","Cell A2 is not less than cell A3")`.

Change the contents of cells A2 and A3 and notice what happens to cell A1, depending on the values of cells A2 and A3.

	A	B
1	Cell A2 is less than cell A3	
2		1
3		3
4		

	A	B
1	Cell A2 is not less than cell A3	
2		5
3		4
4		

Task 4

Without using Excel complete the table based on the screenshot. Then, use Excel to check your answers.

	A	B	C	D
1	12	1	5	12
2	1	11	1	10
3	7	16	3	14
4	2	7	13	8
5	19	20	5	3

Conditional Statement	Displayed Value
<code>=IF(B1=C1,A4,A3)</code>	7
<code>=IF(C2<B4,"Yes","No")</code>	
<code>=IF(A4=2,"Two","Not two")</code>	
<code>=IF(D5*C3>C5+1,B3,2*D1)</code>	
<code>=IF(SUM(A1:A5)<100,SQRT(4),SQRT(9))</code>	
<code>=IF(AVERAGE(A4:D4)>AVERAGE(B3:D4),0,1)</code>	

Creating Graphs

Enter this following into a new worksheet:

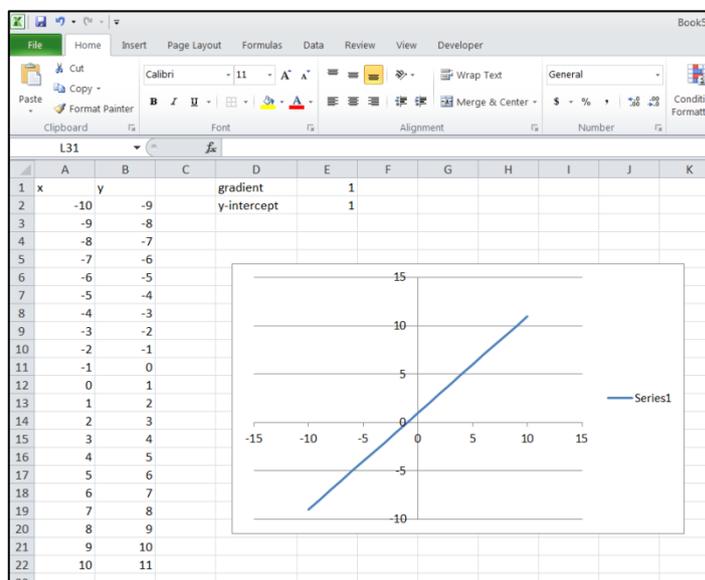
	A	B	C	D	E
1	x	y		gradient	1
2	-10	=E\$1*A2+E\$2		y-intercept	1
3	=A2+1				
4					
5					

Copy cell A3 and paste into cells A4 to A22. Copy cell B2 and paste in cells B3 to B22.

	A	B	C	D	E
1	x	y		gradient	1
2	-10	-9		y-intercept	1
3	-9				
4	-8				
5	-7				
6	-6				
7	-5				
8	-4				
9	-3				
10	-2				
11	-1				
12	0				
13	1				
14	2				
15	3				
16	4				
17	5				
18	6				
19	7				
20	8				
21	9				
22	10				

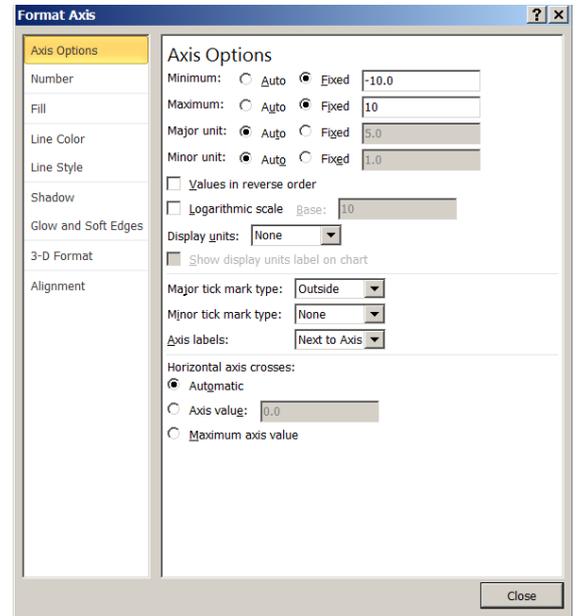
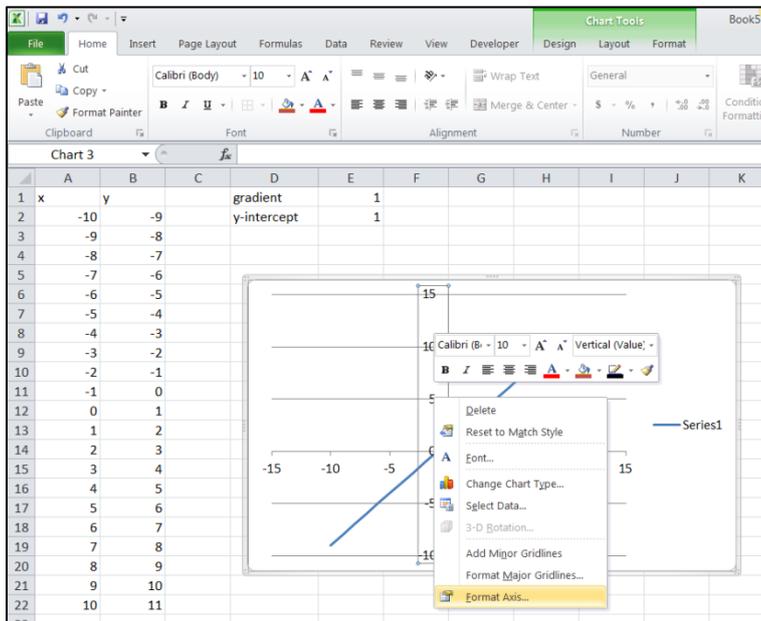
	A	B	C	D	E	F
1	x	y		gradient		1
2	-10	-9		y-intercept		1
3	-9	-8				
4	-8	-7				
5	-7	-6				
6	-6	-5				
7	-5	-4				
8	-4	-3				
9	-3	-2				
10	-2	-1				
11	-1	0				
12	0	1				
13	1	2				
14	2	3				
15	3	4				
16	4	5				
17	5	6				
18	6	7				
19	7	8				
20	8	9				
21	9	10				
22	10	11				

Highlight cells A2 to B22. Click *Insert > Chart > Scatter with Straight Lines*.

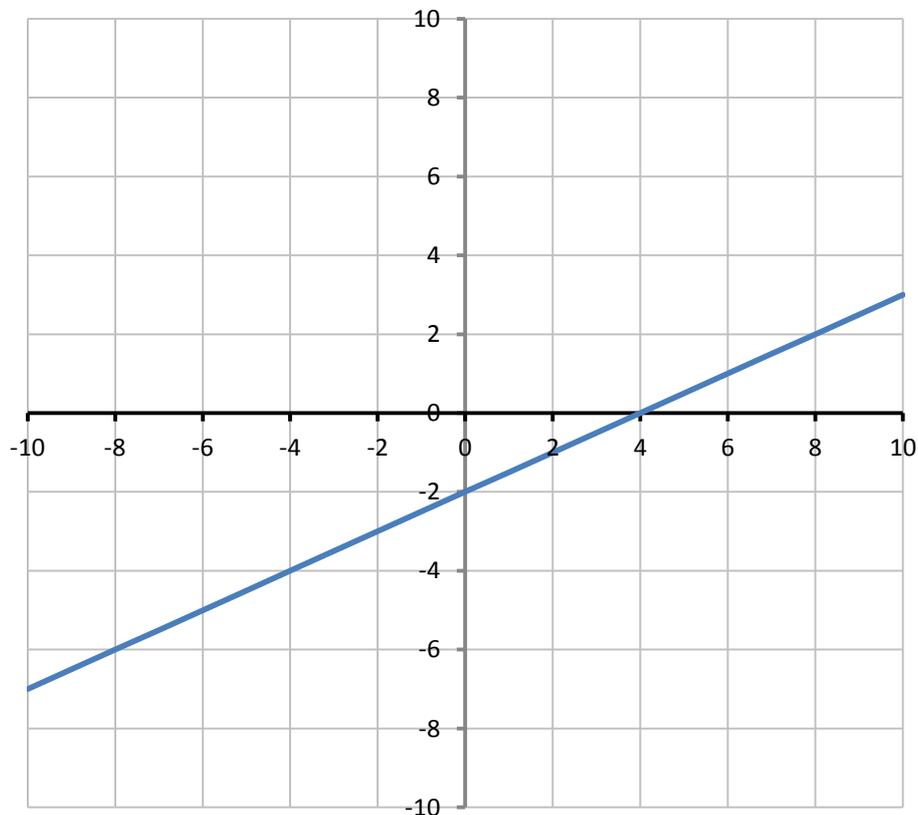


Now, change the values of cells E1 and E2. This will change the gradient and the y-intercept of the graph. However, you may notice that Excel rescales the axes to suit the graph. Sometimes we do not want this to happen.

Right-click on the y-axis and select *Format Axis*. Set the minimum and maximum value of y to -10 and 10 respectively. Right click on the x-axis and also set the minimum and maximum value of x to -10 and 10 respectively. Now, when you change the value of the gradient and y-intercept you can easily see the affect it has on the graph.



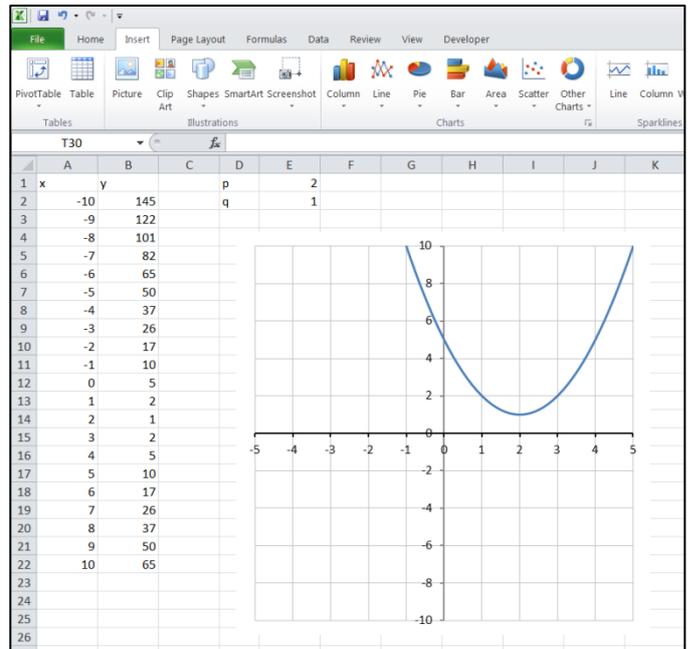
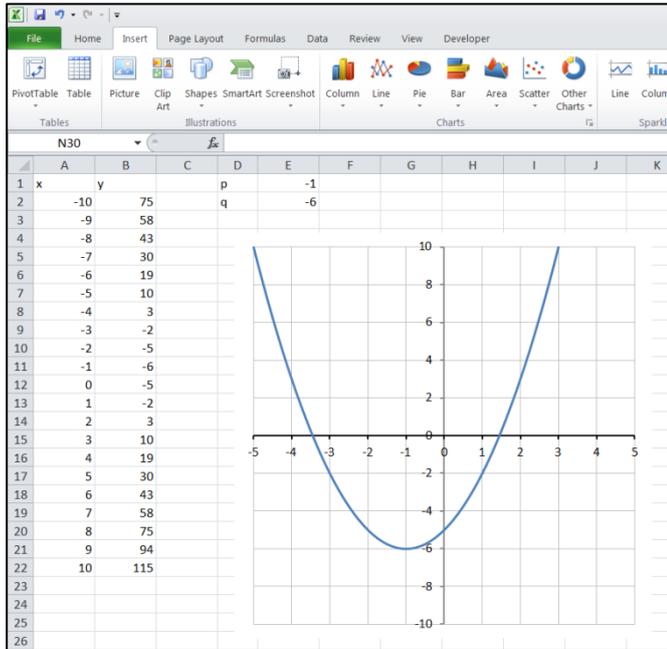
Make the graph more attractive by clicking on it and selecting relevant options from the *Layout* tab.



Task 5

Plot the graph of $y = (x - p)^2 + q$ where the values of p and q are each stored in a cell. Use integer values from -10 to 10 for the x -coordinates. Fix the minimum and maximum x -value on the x -axis to -5 and 5 respectively. Fix the minimum and maximum y -value of the y -axis to -10 and 10 respectively. The chart type should be *Scatter with Smooth Lines*.

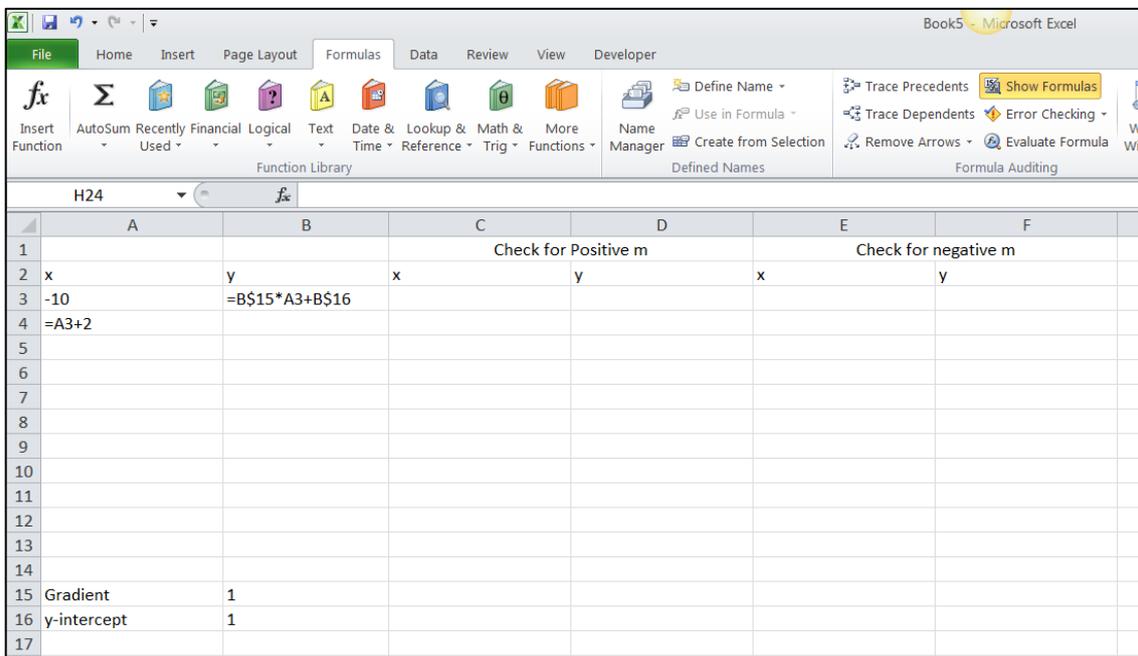
Use the screenshots below to determine the accuracy of your spreadsheet (your gridlines may look different).



Conditional Formatting Graphs

Sometimes we may wish to change the style of a graph depending on certain conditions. For example, suppose we wished to plot the equation $y = mx + c$ in red if the value of m is positive, and in blue if the value of m is not positive.

Set the following up in Excel (note that cells C1 and D1 are merged, as are E1 and F1):



Copy cell A4 and paste it into cells A5 to A13. Copy cell B3 and paste into cells B4 to B13. You have now created the table of values for $y = mx + c$.

We will not use these values to plot the graph; we will use the next four columns. Columns C and D will contain coordinates of points we wish to plot in red, and columns E and F will contain coordinates of points we wish to plot in blue.

We only want to plot red points if the gradient is positive and we only want to plot blue points if the gradient is not positive.

This means we have to use a conditional statement to determine the value of the gradient.

	A	B	C	D	E	F
1			Check for Positive m	Check for negative m		
2	x	y	x	y	x	y
3		-10	-9			
4		-8	-7			
5		-6	-5			
6		-4	-3			
7		-2	-1			
8		0	1			
9		2	3			
10		4	5			
11		6	7			
12		8	9			
13		10	11			
14						
15	Gradient		1			
16	y-intercept		1			
17						

In cell C3 enter `=IF(B15>0,A3,-1000)`. This means that if the value of the gradient is positive, the x -coordinate for that point will appear in the cell. If it is not positive then -1000 will appear in the cell. If we fix the scales on the axes of our graph we can make sure that the point will not be visible if it is equal to -1000.

Copy cell C3 and paste into cell D3. Copy cells C3 and D3 and paste into cells C4 to D13:

	A	B	C	D	E
1			Check for Positive m	Check for n	
2	x	y	x	y	x
3		-10	-9	-1000	
4		-8	-7		
5		-6	-5		
6		-4	-3		
7		-2	-1		
8		0	1		
9		2	3		
10		4	5		
11		6	7		
12		8	9		
13		10	11		
14					
15	Gradient		1		
16	y-intercept		1		
17					

	A	B	C	D	E
1			Check for Positive m	Check for	
2	x	y	x	y	x
3		-10	-9	-1000	-1000
4		-8	-7	-8	-7
5		-6	-5	-6	-5
6		-4	-3	-4	-3
7		-2	-1	-2	-1
8		0	1	0	1
9		2	3	2	3
10		4	5	4	5
11		6	7	6	7
12		8	9	8	9
13		10	11	10	11
14					
15	Gradient		1		
16	y-intercept		1		
17					

In cell E3 enter `=IF(B15>0,-1000,A3)`. This means that if the value of the gradient is positive, the value of -1000 will appear in the cell. If it is not positive then the x -coordinate for that point will appear in the cell.

Copy cell E3 and paste into cell F3. Copy cells E3 and F3 and paste into cells E4 to F13:

Function Library						
E3						
=IF(\$B\$15>0,-1000,A3)						
	A	B	C	D	E	F
1			Check for Positive m		Check for negative m	
2	x	y	x	y	x	y
3		-10	-9	-10	-9	-1000
4		-8	-7	-8	-7	
5		-6	-5	-6	-5	
6		-4	-3	-4	-3	
7		-2	-1	-2	-1	
8		0	1	0	1	
9		2	3	2	3	
10		4	5	4	5	
11		6	7	6	7	
12		8	9	8	9	
13		10	11	10	11	
14						
15	Gradient		1			
16	y-intercept		1			
17						

Function Library						
H25						
=IF(\$B\$15>0,-1000,A3)						
	A	B	C	D	E	F
1			Check for Positive m		Check for negative m	
2	x	y	x	y	x	y
3		-10	-9	-10	-9	-1000
4		-8	-7	-8	-7	-1000
5		-6	-5	-6	-5	-1000
6		-4	-3	-4	-3	-1000
7		-2	-1	-2	-1	-1000
8		0	1	0	1	-1000
9		2	3	2	3	-1000
10		4	5	4	5	-1000
11		6	7	6	7	-1000
12		8	9	8	9	-1000
13		10	11	10	11	-1000
14						
15	Gradient		1			
16	y-intercept		1			
17						

Now, change the gradient to a negative value and the values in columns C, D, E and F should change.

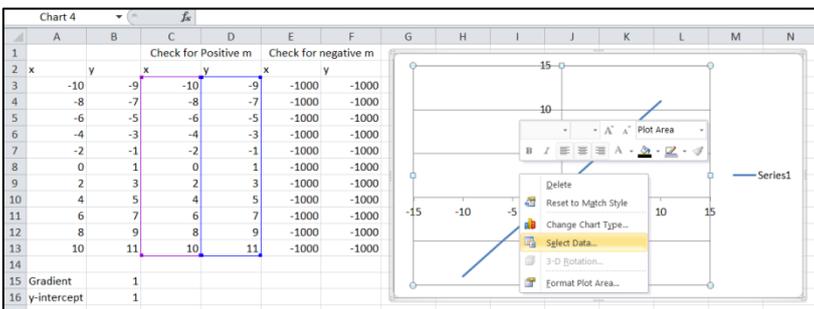
Change the gradient back to a positive value.

We now need to plot this data. Highlight cells C3 to D13 and click *Insert > Chart > Scatter with Straight Lines*.

Right click on the graph and click *Select Data...*

From the *Select Data* menu click *Add*.

Function Library						
L27						
=IF(\$B\$15>0,-1000,A3)						
	A	B	C	D	E	F
1			Check for Positive m		Check for negative m	
2	x	y	x	y	x	y
3		-10	21	-1000	-1000	-10
4		-8	17	-1000	-1000	-8
5		-6	13	-1000	-1000	-6
6		-4	9	-1000	-1000	-4
7		-2	5	-1000	-1000	-2
8		0	1	-1000	-1000	0
9		2	-3	-1000	-1000	2
10		4	-7	-1000	-1000	4
11		6	-11	-1000	-1000	6
12		8	-15	-1000	-1000	8
13		10	-19	-1000	-1000	10
14						
15	Gradient		-2			
16	y-intercept		1			
17						

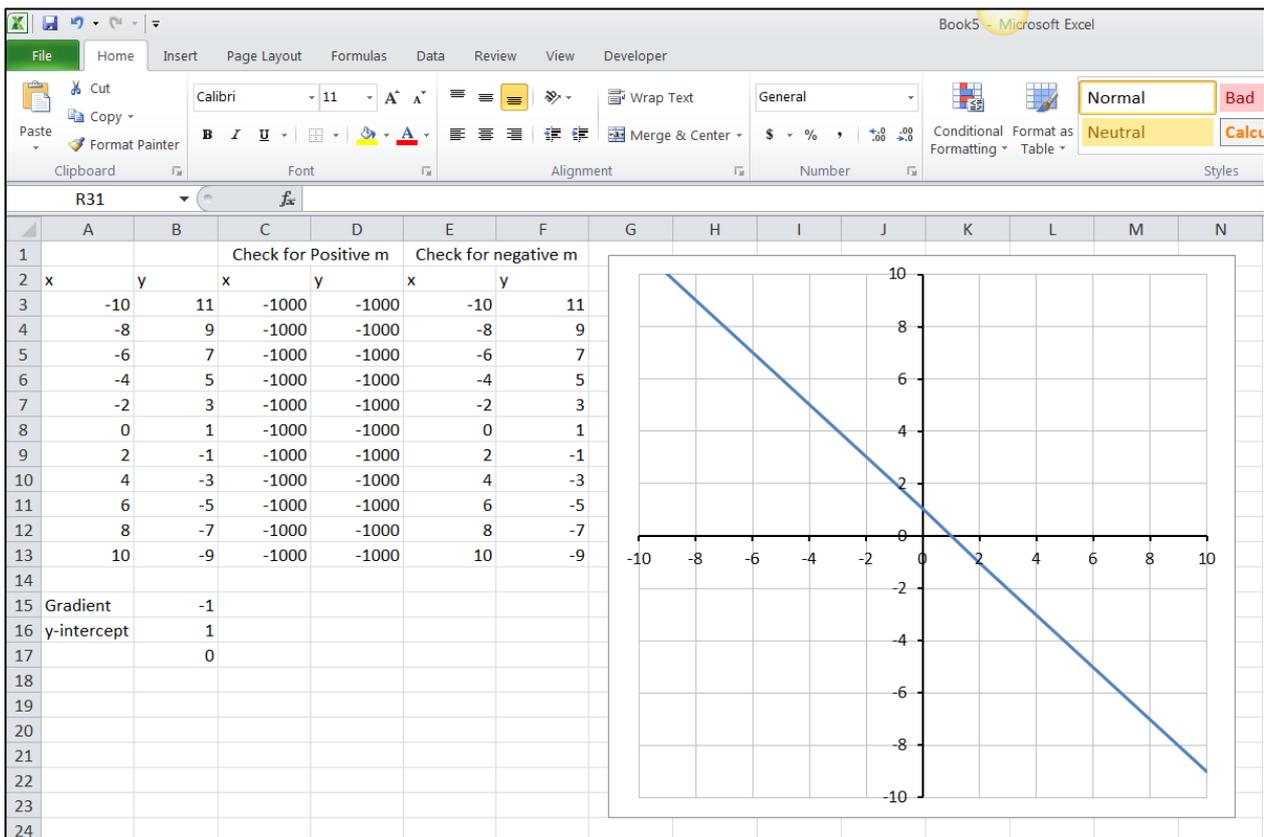
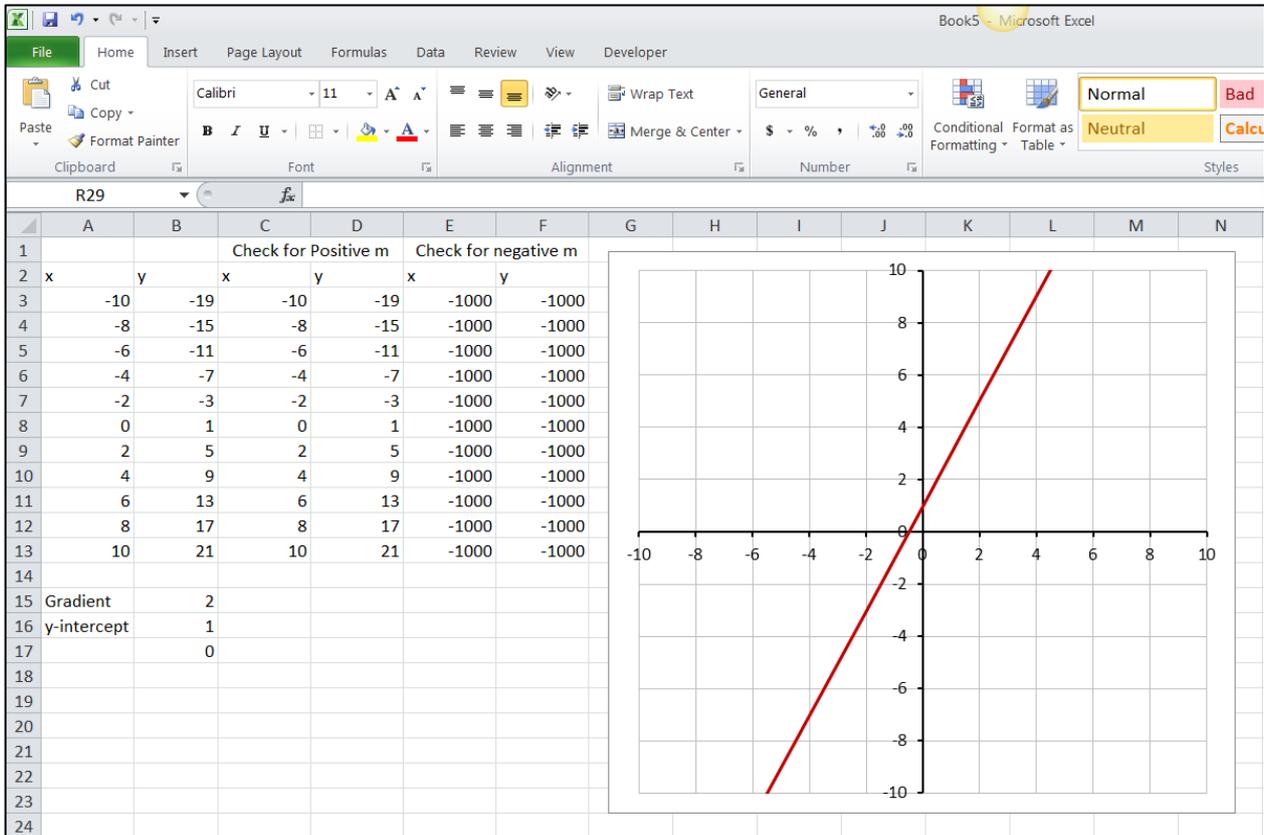


Place the cursor in the *Series X Values* box and then highlight cells E3 to E13.

Place the cursor in the *Series Y Values* box, delete the existing contents, and then highlight cells F3 to F13.

The contents of these two boxes should now look like the screenshot on the right.

Click *Okay* twice and your graph will be updated. However, you will need to fix the range of your axes to see it clearly. Once you have fixed your axes, and formatted your graph to make it more attractive you should have something similar to the following. Note that you may need to change the colour of your lines to meet the requirements of this example. You can do this by right-clicking on your lines and selecting *Format Data Series...* and changing the line colour.



Task 6

Plot 50 horizontal lines of length 1. The position of each line should be randomly determined (set suitable boundaries so they are reasonably close together. Lines which start in the first quadrant should be red, lines which start in the second quadrant should be green, lines which start in the third quadrant should be blue and lines which start in the fourth quadrant should be orange.

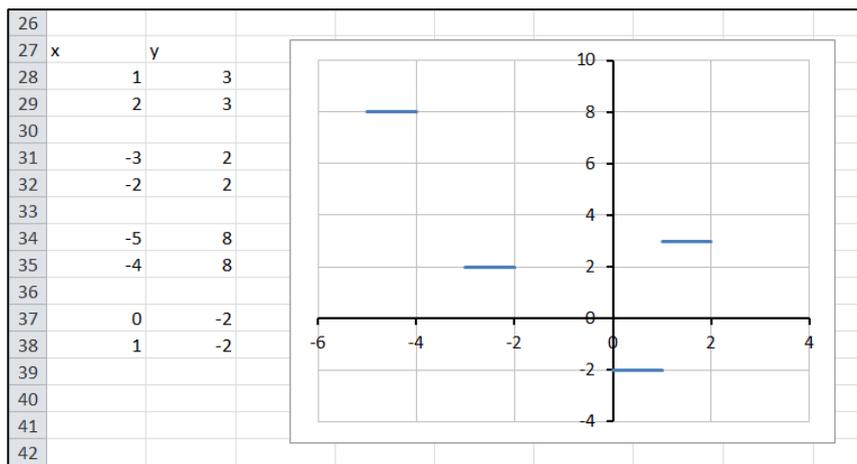
Hints

- The IF function can test multiple conditions by combining it with AND. For example

$$=IF(AND(A1<19,A1>12),3,7)$$

will display the value of 3 if A1 is between 12 and 19, and the value of 7 otherwise.

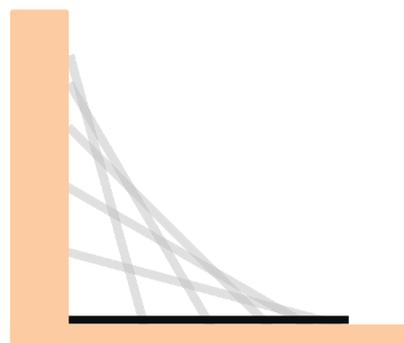
- The RANDBETWEEN function only accepts non-negative integers. If you need decimals you need to divide the random number by something. If you need negative numbers you need to subtract something from the random number.
- To plot separate line segments, leave a space between each set of coordinates



Appendix B: Geogebra Tutorial

Consider the following problem:

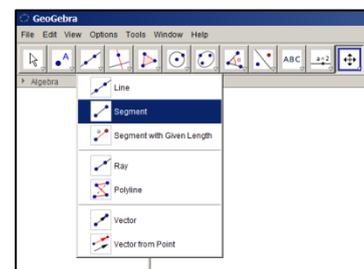
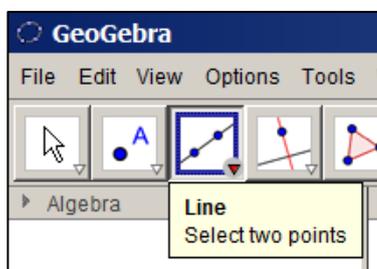
A 10 m long ladder is leaning against a wall. A man is standing exactly half way up the ladder. The ladder slips so that the top of the ladder slides down the wall, and the bottom of the ladder slides along the floor, eventually landing horizontally (shown in the diagram on the right). If the floor and the wall represent the x and the y -axis respectively, determine the equation of the path that the man travels.



Drawing Line Segments

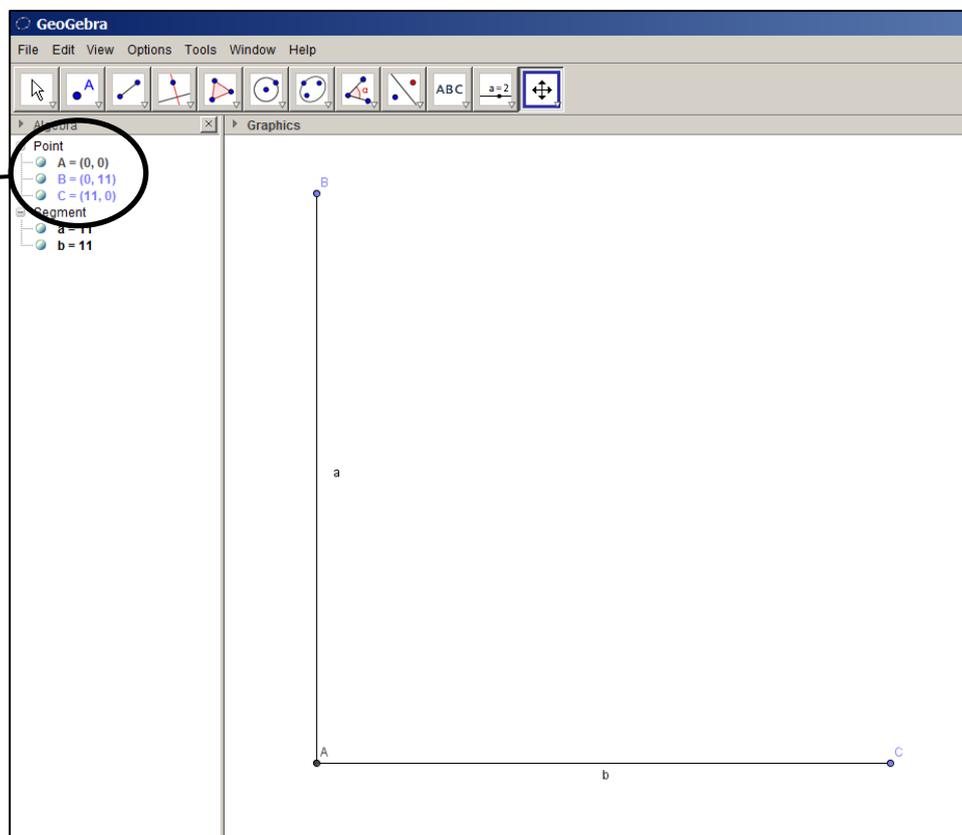
Select the segment tool. If you cannot see it, click on the white arrow on the third button to reveal more functions.

Draw a segment from $(0,0)$ to $(0,11)$. This will represent the wall. Draw a segment from $(0,0)$ to $(11,0)$. This will represent the floor.



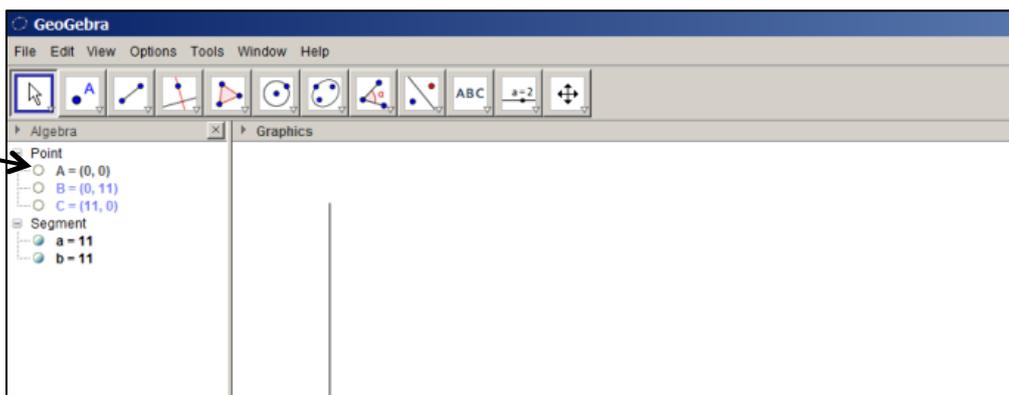
Right click on the screen and turn off the *Axes*:

If you didn't create your points accurately you can type the exact values that you want here.



The two lines are labelled with the letters a and b . If we right-click on each line you can hide these labels. It is not necessary to display points A, B and C. If we click the green dots next to each point in the *Algebra* box it will hide the points:

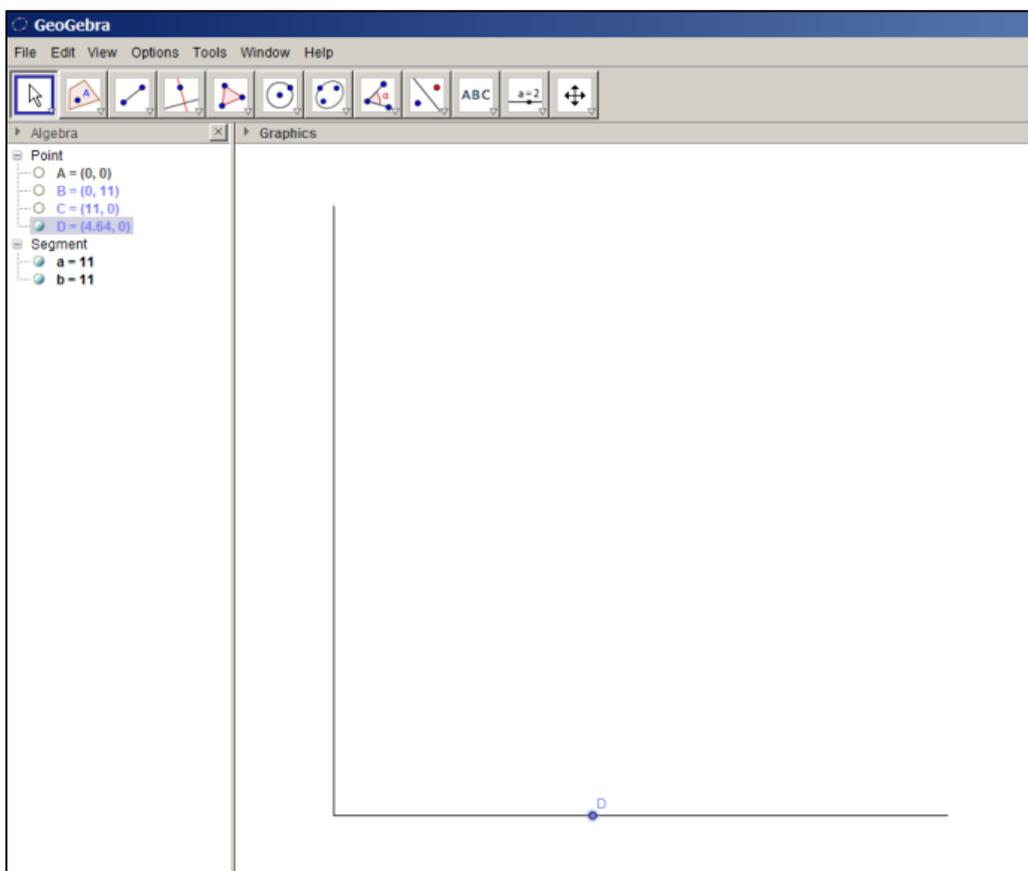
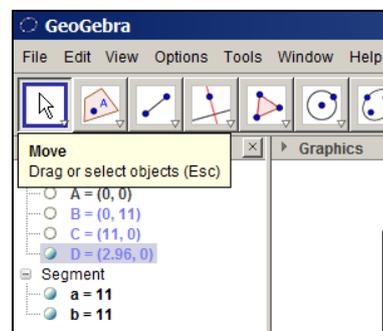
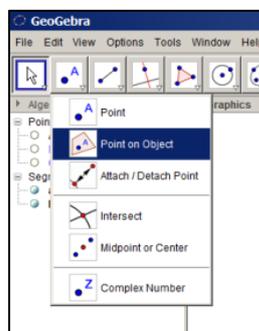
Hide the points by clicking on the green dots.



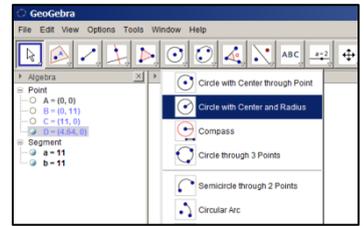
Restricting a Point to a Line

The base of the ladder is restricted to the floor, so we need to create a point, which represents the base of the ladder, that is restricted to the horizontal line. Select the *Point on Object* tool and place a point on the horizontal line.

If we try and move the point, using the *Move* tool we will see that it is restricted to the floor.

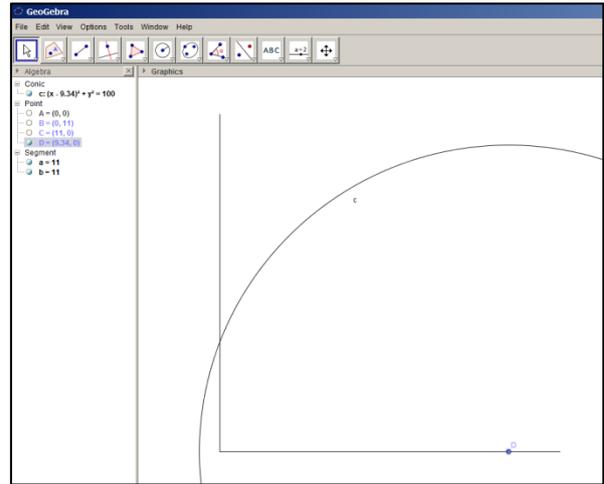
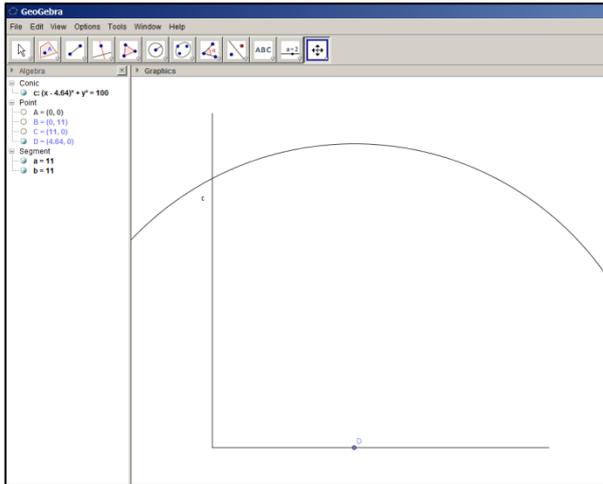


The top of the ladder is restricted to the wall. However, we cannot simply create another random point on the wall. The ladder is 10 m in length so the point on the wall must always be at a distance of 10 from the point on the floor.



Drawing Circles

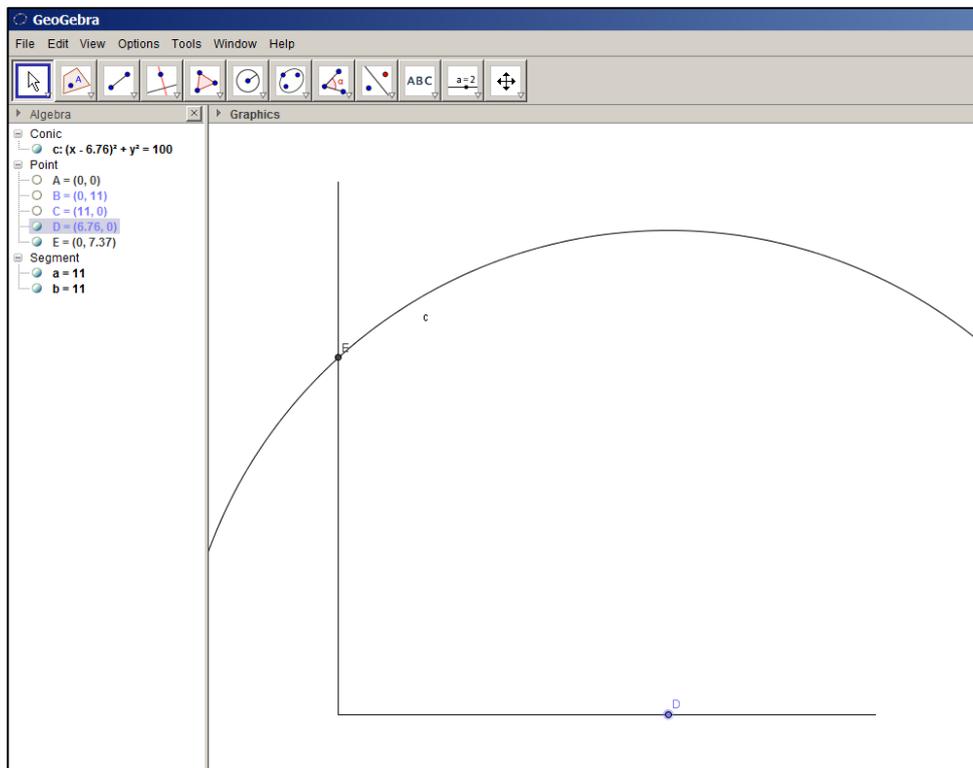
Select the *Circle with Centre and Radius* tool (note that there are several tools to create a circle). Draw a circle centred at point D with radius 10. Now, try moving point D. Notice that the circle will always move with it.



The point where the circle intersects the vertical line represents the end of the ladder (it will always be 10 m from point D).

Select the *Point on Object* tool and create a new point at the intersection of the circle and the wall.

Test to see if it has been created correctly by moving point D. Our new point should always move to where the circle intersects the wall.

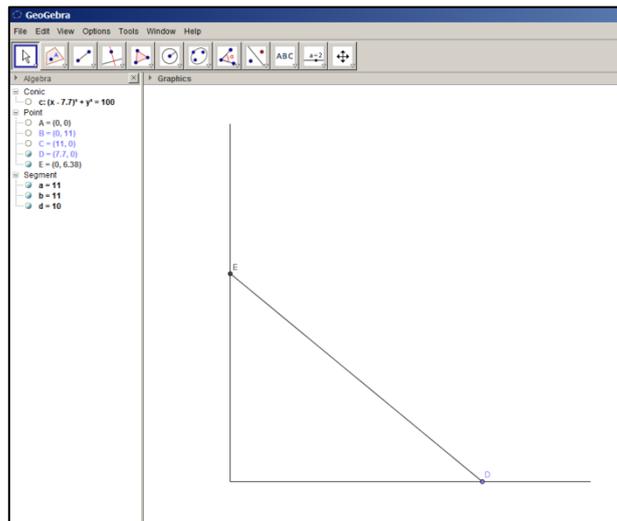


Now we can hide the circle. A quick way to do this is by clicking the green dot next to the equation of the circle in the *Conic* section of the *Algebra* window.

Draw a line segment from point D to point E. This will represent the ladder. Move point D away from the wall and the ladder should move (slip) down the wall.

The ladder is labelled with a letter *d*. This can be hidden by right-clicking on the ladder and unchecking *Show Label*.

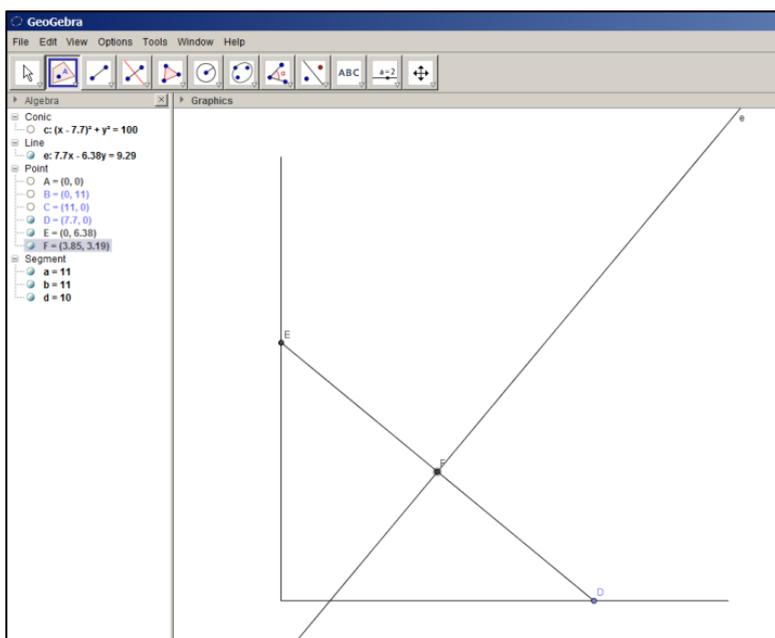
We now need to create a point that represents the man standing on the ladder, half way along its length.



Perpendicular Bisectors

Locate and select the *Perpendicular Bisector* tool and click on the ladder.

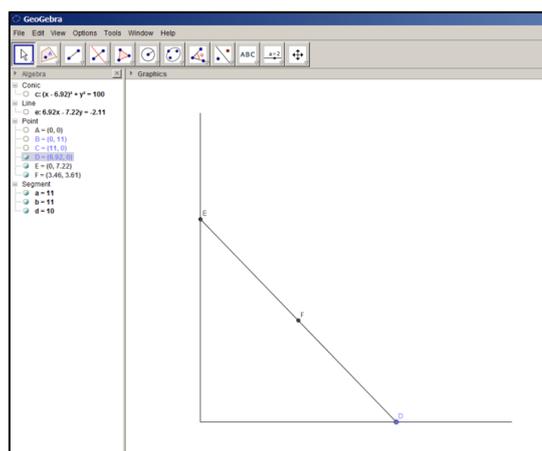
Select the *Point on Object* tool and create a new point where the perpendicular bisector intersects the ladder:



We can hide the perpendicular bisector and the model is complete.

Move point D to see the path of the man as the ladder slips.

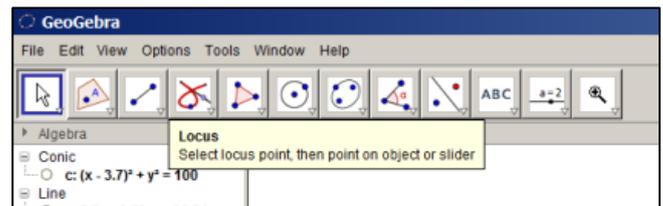
We now need to use the model to determine the equation of the path of the man.



Loci

We can use Geogebra to draw the path of the man as the ladder slides down the wall.

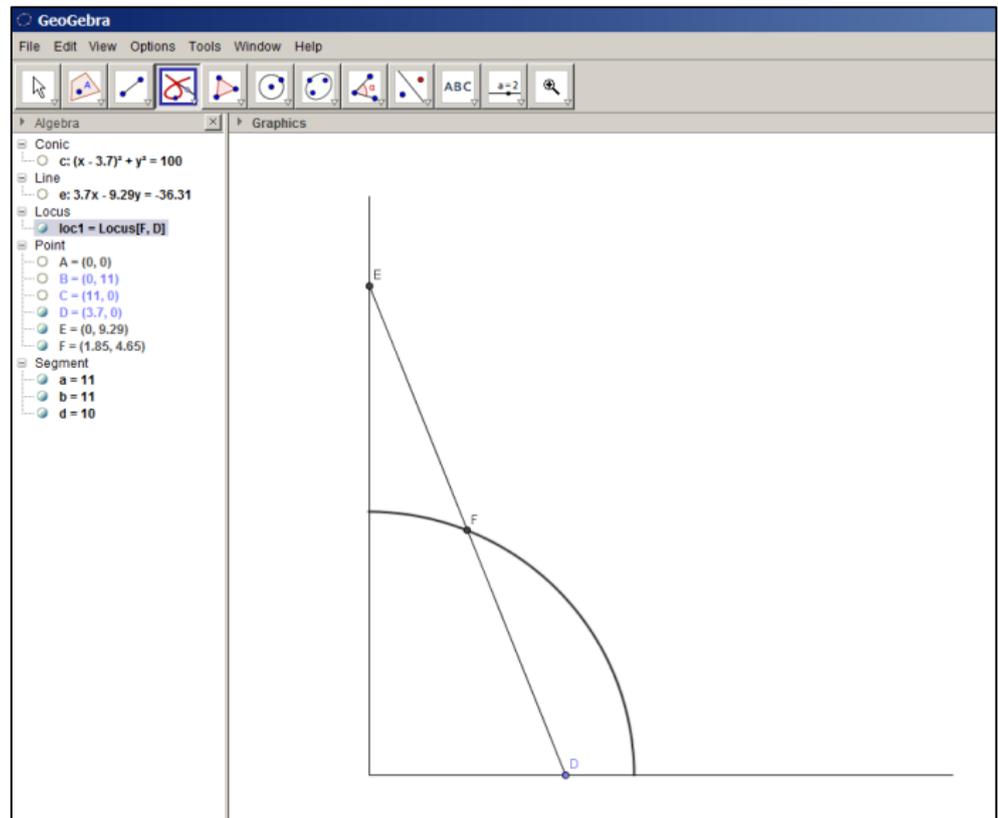
Locate and select the *Locus* tool. Notice that if you hold the mouse over the tool it will give you a description of how to use it.



Click on the man (point F) and then click on the base of the ladder (point D). The path of the man will then be drawn onto the screen.

The path appears to be a quarter of a circle, centred at the origin.

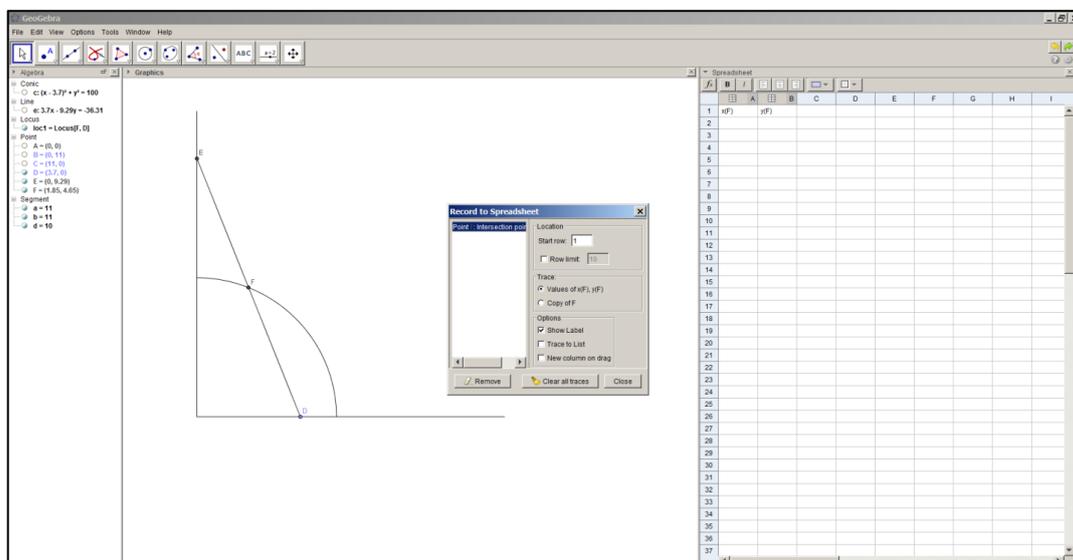
There is another method we can use to confirm our theory that the path of the man is circular. Geogebra contains a built in spreadsheet that can record the coordinates of a point.



Recording to the Spreadsheet

Select *View > Spreadsheet* and the spreadsheet will appear on the right of the screen.

Right-click on point F and select *Record to Spreadsheet*. Then click *Close* to close the pop-up window.



Now, move point D and its coordinates will be recorded to the spreadsheet.

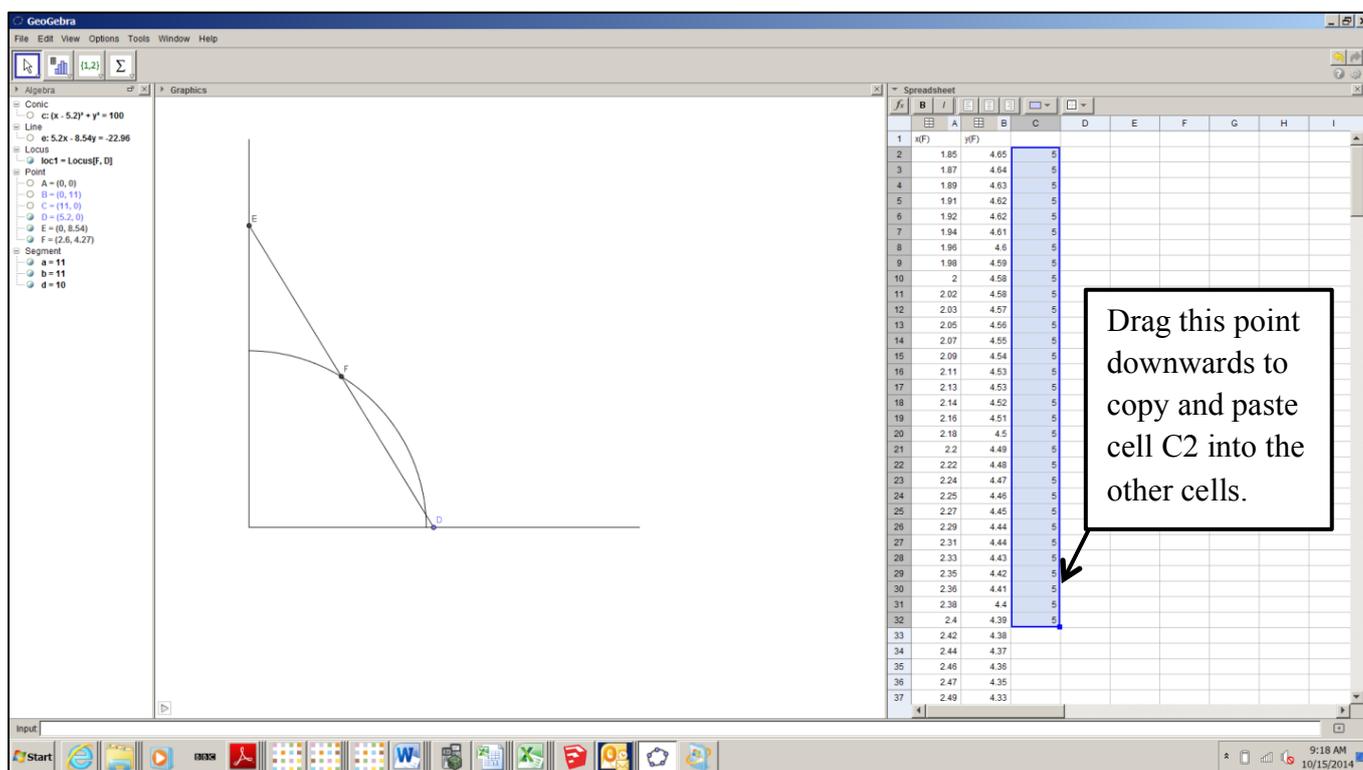
Alternatively, right-click point D and select *Animation On* and it will automatically move. To pause the animation press the pause button in the bottom left corner.

Our spreadsheet will now contain many coordinates representing the various positions of F. If the path of F is indeed circular, centred at the origin, then it will always be a fixed distance from the origin. We can check this by using the Pythagorean theorem.

In cell C2 type: $=\text{sqrt}(A2^2+B2^2)$

This calculates the value of $\sqrt{x^2 + y^2}$, which is the distance from the origin.

If we copy and paste this formula into all of the other rows by dragging the bottom right corner of the cell downwards, we can see that the distance of point D from the origin is always equal to 5.



The position of the man can be given by the equation:

$$y = \sqrt{25 - x^2}$$

Note that we haven't actually proven that this equation is true; although the evidence given by Geogebra strongly hints that it is true. To prove it we have to use algebra.

Proof:

The coordinates of the base of the ladder are $(t, 0)$ for some $t \in [0, 10]$.

The coordinates of the top of the ladder are therefore $(0, \sqrt{100 - t^2})$.

The coordinates of the man are therefore $(\frac{t}{2}, \frac{\sqrt{100 - t^2}}{2})$.

So we have:

$$x = \frac{t}{2} \Rightarrow t = 2x$$

$$\begin{aligned} y &= \frac{\sqrt{100 - t^2}}{2} \\ &= \frac{\sqrt{100 - 4x^2}}{2} \\ &= \sqrt{25 - x^2} \end{aligned}$$

This completes the proof.

Appendix C: The Definite Integral

Suppose we wish to approximate the shape of the curve $y = g(x)$ on the interval $[a, b]$ using the following method:

- Let $x_1 = a$ and point P_1 have coordinates $(x_1, g(x_1))$.
- Let point P_2 have an x -coordinate of $x_2 = x_1 + \Delta x$. Draw a straight line from P_1 to P_2 with a gradient equal to $g'(x_1)$.
- Let point P_3 have an x -coordinate of $x_3 = x_2 + \Delta x$. Draw a straight line from P_2 to P_3 with a gradient equal to $g'(x_2)$.
- Repeat this process until point P_i has an x -coordinate equal to b .

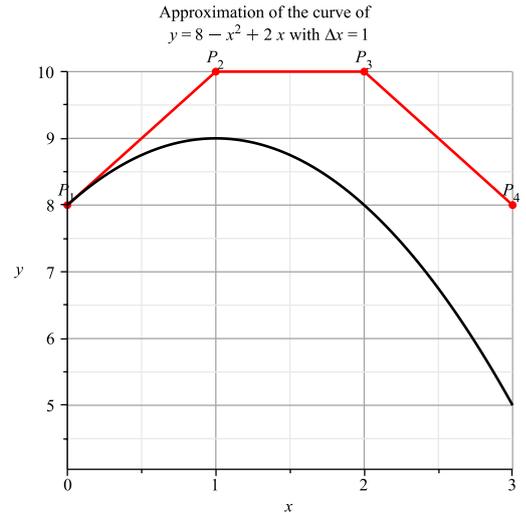


Figure 1

Figures 1, 2 and 3 show the results of this process for decreasing values of Δx for the function $y = 8 - x^2 + 2x$ on the interval $[0, 3]$.

As the value of Δx becomes smaller, the shape of the red approximation approaches the shape of the function.

1. Complete the following:

$$g(b) = g(a) + \lim_{n \rightarrow \infty} \sum_{i=1}^n \dots\dots\dots$$

2. Let $f(x) = g'(x)$. Rewrite the equation in terms of $f(x)$ and $F(x)$, where $F(x)$ is the antiderivative of $f(x)$.

3. Introduce the notation $\int_a^b f(x)dx = F(b) - F(a)$ and show that

$$\int_a^b f(x)dx = \lim_{n \rightarrow \infty} \sum_{i=1}^n f(x_i)\Delta x$$

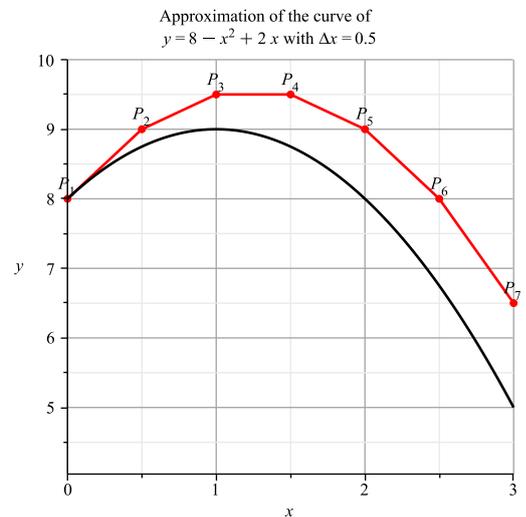


Figure 2

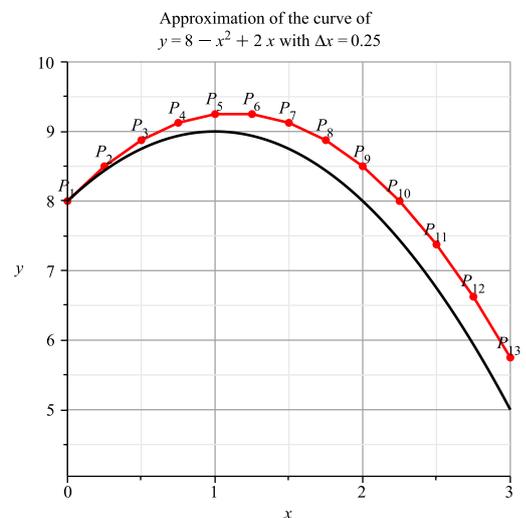


Figure 3