

Projectile motion – Mathematics IA

Introduction

Projectile motion is the motion of an object that is moving in air and experiences the force of gravity.¹ My interest in the topic of projectile motion stemmed from the fact that my favorite topic in physics class was kinematics. However, I realized that the motion we considered in class was limited in the way that it was usually only one-dimensional and did not take factors such as air resistance into account. Therefore, the aim of this investigation is to understand the motion in air more fully, including when it is two-dimensional, and then taking air resistance into account.

Defining symbols:

t: time (s)

s: displacement (m)

a: acceleration (ms^{-2})

g: acceleration due to gravity (On Earth $-9.8 ms^{-2}$)

u: initial velocity (ms^{-1})

u_x: initial horizontal velocity (ms^{-1})

u_y: initial vertical velocity (ms^{-1})

θ: angle of initial velocity to horizontal (degrees)

v: instantaneous velocity (ms^{-1})

v_x: instantaneous horizontal velocity (ms^{-1})

v_y: instantaneous vertical velocity (ms^{-1})

α: angle of instantaneous velocity to horizontal (degrees)

m: mass (kg)

k: a constant

In a vacuum

One-dimensional motion

In a one-dimensional vacuum, if a ball is thrown vertically and caught at the same position, equations for the motion of the ball over time can be modeled simply.

The displacement (s) and time (t) have the relationship:

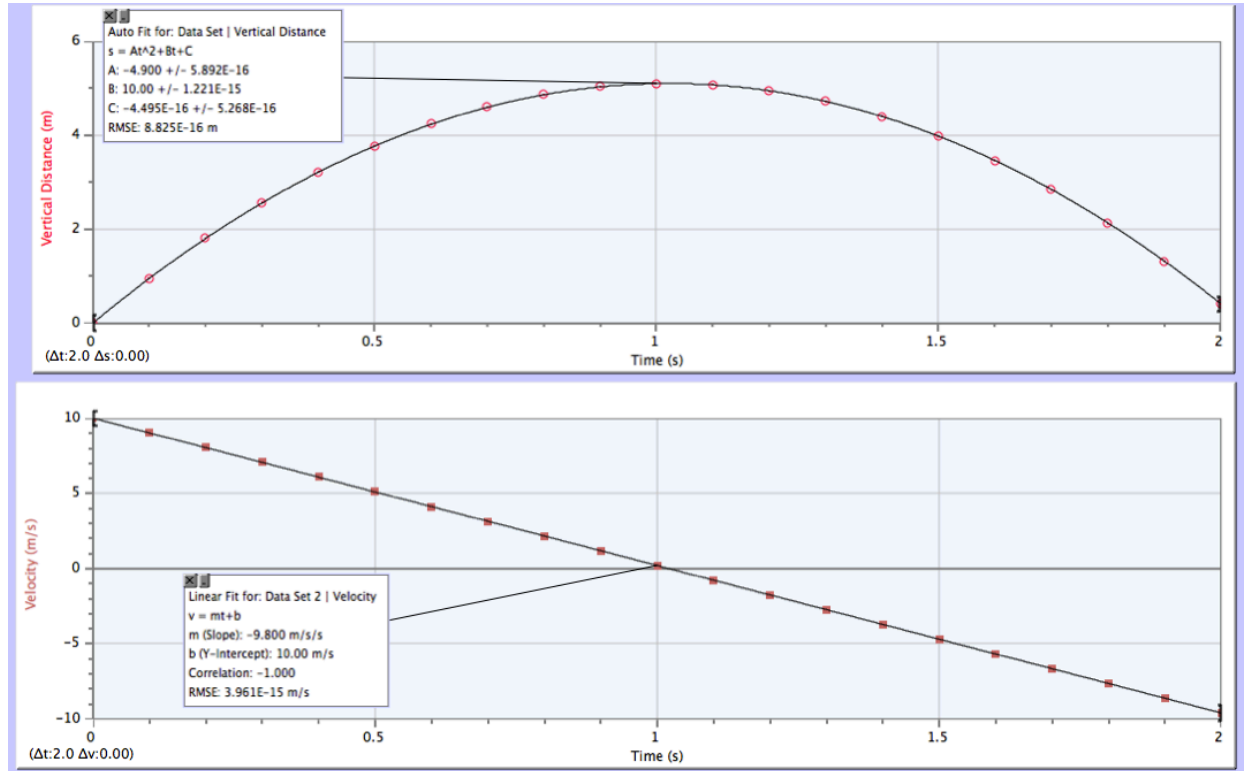
$$s = \frac{1}{2}at^2 + ut \quad (\text{where } a=\text{acceleration, and } u=\text{initial velocity})$$

Velocity is the rate of change of displacement, and therefore is represented by:

¹ The Physics Classroom. (2015). *What is a projectile?* Retrieved February 16, 2015 from the Physics Classroom:
<http://www.physicsclassroom.com/class/vectors/Lesson-2/What-is-a-Projectile>

$$v = \frac{ds}{dt} = at + u$$

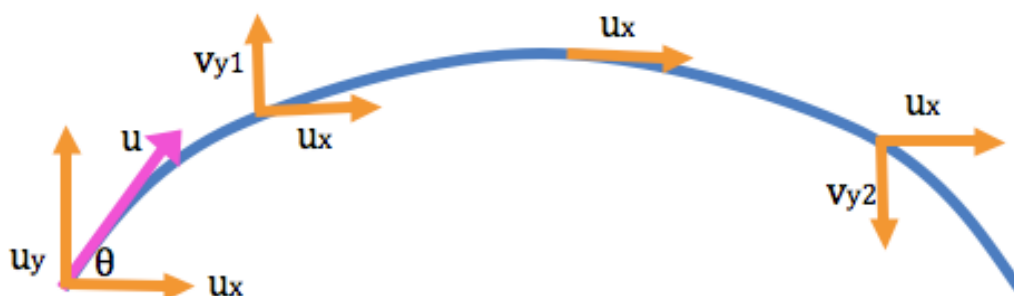
The graphs below show the vertical distance and velocity of a ball that is thrown vertically with an initial speed of 10 ms^{-1} . (Graphs created using LoggerPro)



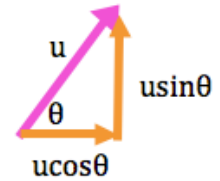
The gradient of the velocity represents the acceleration. As the only force acting on the object is the acceleration due to gravity, the gradient is equal to -9.8 ms^{-2} (assuming we are on Earth).

Two-dimensional motion

When an object is thrown at an angle to the ground, it experiences projectile motion. In a vacuum, the only force acting on the object would be the downward gravitational force. Therefore the horizontal velocity (u_x) stays constant while vertical velocity ($u_y, v_{y1}, v_{y2},$) decelerates and then accelerates downwards. The resulting shape of the path of the object will be a parabola.



Using trigonometry, the initial vertical and horizontal velocities are:
 $u_y = u \sin \theta$ and $u_x = u \cos \theta$



For example, if the initial velocity was 10 ms^{-1} at a 30° angle to the ground:

The vertical component of the initial velocity would be $10 \times \frac{1}{2} = 5 \text{ ms}^{-1}$, and the horizontal component would be $10 \times \frac{\sqrt{3}}{2} = 8.66 \text{ ms}^{-1}$.

To find the displacement, the same equation $s = \frac{1}{2}at^2 + ut$ is used, but the vertical and horizontal displacements must be thought of separately. Let x represent the horizontal displacement, and y represent the vertical displacement.

There is no horizontal force on the object once it is in the air, so $a_x = 0$
 Thus, $x = u_x t$ or $x = u \cos \theta t$

The acceleration due to gravity works vertically, and therefore $a_y = g$

$$y = \frac{1}{2}gt^2 + u_y t \text{ or } y = \frac{1}{2}gt^2 + u \sin \theta t$$

Finding the time the object will be in the air:

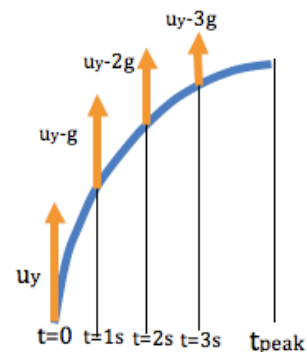
Only the vertical velocity will affect the time that the object is in the air, as the vertical displacement must be above ground for it to be in the air.

The object will have a vertical velocity of 0 ms^{-1} when it is at the highest point, as it stops rising starts falling.

The acceleration due to gravity is -9.8 ms^{-2} . If only the time it takes for the object to reach its highest point is considered, it reduces the vertical velocity of the object by 9.8 m/s every second.

Therefore, by calculating how many times 9.8 goes into u_y , the time taken for the object to reach a maximum height can be calculated. As $g = -9.8$, $-g$ must be used to express 9.8 .

$$t_{\text{peak}} = \frac{u_y}{-g} = \frac{u \sin \theta}{-g}$$



As the path is symmetrical, it will take the same amount of time for the object to reach the highest point, and to come back to the starting height. Therefore, the total air time of the object will be twice that of the time it takes for the object to reach its peak height.

$$t_{\text{air}} = 2t_{\text{peak}} = \frac{2u \sin \theta}{-g}$$

Finding the distance the object will travel:

Distance=speed×time

As the distance is the horizontal distance, only the horizontal velocity (which is constantly u_x) is affects it.

$$D = u_x \times t_{air}$$

$$D = u \cos \theta \times \frac{2u \sin \theta}{g}$$

$$D = \frac{2u^2}{g} \cos \theta \sin \theta$$

Finding the optimal angle for the maximum distance:

$$D = \frac{2u^2}{-g} \cos \theta \sin \theta$$

$$\frac{dD}{d\theta} = \frac{2u^2}{-g} (-\sin \theta \sin \theta + \cos \theta \cos \theta)$$

$$\frac{dD}{d\theta} = \frac{2u^2}{-g} (-\sin^2 \theta + \cos^2 \theta)$$

$$\frac{dD}{d\theta} = 0 \text{ for the maximum}$$

$$0 \leq \theta \leq 90$$

$$0 = \frac{2u^2}{-g} (-\sin^2 \theta + \cos^2 \theta)$$

$$0 = -\sin^2 \theta + \cos^2 \theta$$

$$\sin^2 \theta = \cos^2 \theta$$

$$\frac{\sin^2 \theta}{\cos^2 \theta} = \frac{\cos^2 \theta}{\cos^2 \theta}$$

$$\left(\frac{\sin \theta}{\cos \theta}\right)^2 = 1$$

$$\tan^2 \theta = 1$$

$$\theta = \tan^{-1}(\sqrt{1})$$

$$\theta = 45^\circ$$

Check: (Graphically)

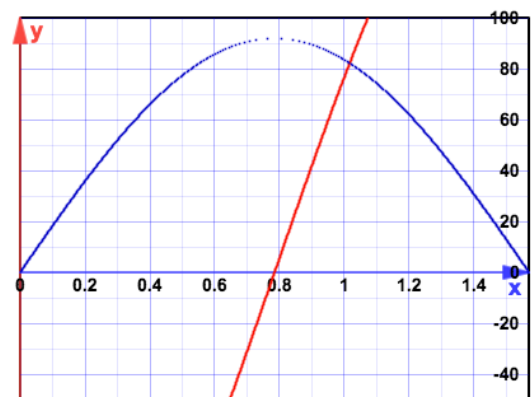
Blue line: horizontal distance traveled.

$$D = \frac{2u^2}{-g} \cos \theta \sin \theta$$

Red line: derivative of horizontal distance.

$$\frac{dD}{d\theta} = \frac{2u^2}{-g} (-\sin^2 \theta + \cos^2 \theta)$$

The red line crosses the x-axis at 0.78radians, which corresponds to 45degrees and therefore confirms my calculations.



Air Resistance

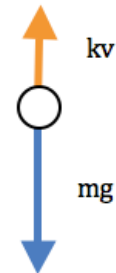
When we toss a ball, shoot an arrow or hit a golf ball we do not do so in a vacuum. Air resistance must be taken into account in order to accurately model the projectile motion.

Air resistance is a force that is proportional and in the opposite direction to the velocity of the object.²

One-dimensional motion

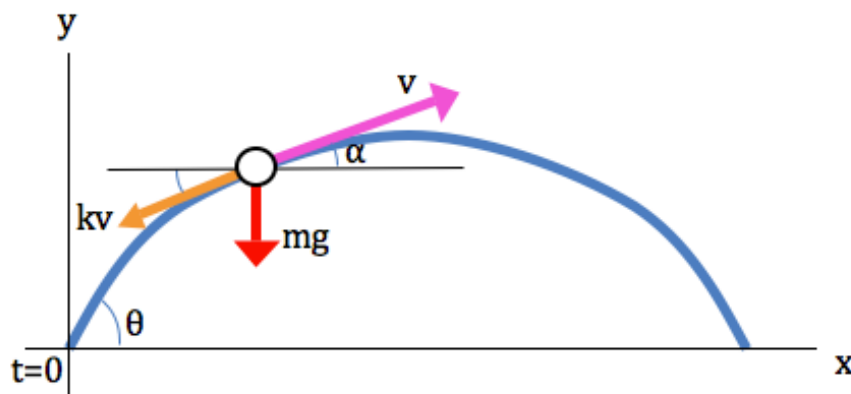
Therefore, if a ball were dropped vertically, the air resistance would be in the upward direction and could be written as kv , where k is a constant.

The second force would be due to gravity, calculated by the product of the mass (m) and the acceleration due to gravity (g). The net force would be $kv - mg$ (where the positive direction is upward).



Two-dimensional motion

To apply this to projectile motion, the horizontal and vertical motion would have to be considered separately.



Horizontal motion

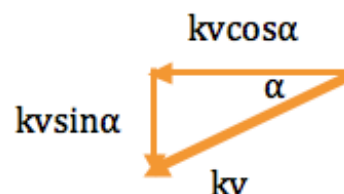
Horizontal force

Only the force due to air resistance is acting on the object.

$F_x = -kv \cos \alpha$ (Negative sign because it is in negative direction on x-axis)

Deriving the horizontal velocity equation

Using Newton's second law, $F = ma$



² The Physics Classroom. (2015). *Types of Forces*. Retrieved February 13, 2015 from The Physics Classroom: <http://www.physicsclassroom.com/class/newtlaws/Lesson-2/Types-of-Forces>

$$-kv \cos \alpha = m \frac{dv_x}{dt}$$

$$-kv_x = m \frac{dv_x}{dt}$$

$$\int \frac{-k}{m} dt = \int \frac{1}{v_x} dv_x$$

$$\frac{-k}{m} t = \ln(v_x) + c$$

Substitute: at $t = 0$, $v_x = u \cos \theta$

$$0 = \ln(u \cos \theta) + c$$

$$c = -\ln(u \cos \theta)$$

Substitute c into previous equation:

$$\frac{-k}{m} t = \ln(v_x) - \ln(u \cos \theta)$$

$$\frac{-k}{m} t = \ln\left(\frac{v_x}{u \cos \theta}\right)$$

$$e^{-\frac{k}{m} t} = \frac{v_x}{u \cos \theta}$$

$$v_x = u \cos \theta e^{-\frac{k}{m} t}$$

Deriving the horizontal displacement equation

Let the horizontal displacement be represented by the x coordinate.

Velocity is the rate of change of displacement. Therefore,

$$x = \int v_x dt = \int u \cos \theta e^{-\frac{k}{m} t} dt$$

$$x = u \cos \theta \int e^{-\frac{k}{m} t} dt$$

$$x = -\frac{m}{k} u \cos \theta e^{-\frac{k}{m} t} + c$$

Substitute: at $t = 0$, $x = 0$

$$0 = -\frac{m}{k} u \cos \theta + c$$

$$c = \frac{m}{k} u \cos \theta$$

Substitute c into previous equation:

$$x = -\frac{m}{k} u \cos \theta e^{-\frac{k}{m}t} + \frac{m}{k} u \cos \theta$$

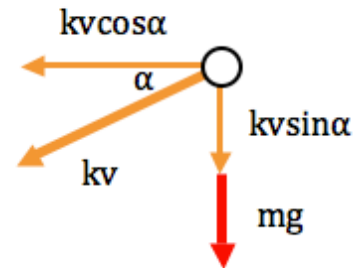
$$x = \frac{m}{k} u \cos \theta (1 - e^{-\frac{k}{m}t})$$

Vertical motion

Vertical force

Both the force due to air resistance and gravity are acting on the vertical motion.

$F_y = -mg - kv \sin \alpha$ (both forces are downward and therefore require a negative sign.)



Deriving the vertical velocity equation

Once again, using Newton's second law $F_y = ma_y$

$$-mg - kv \sin \alpha = m \frac{dv_y}{dt}$$

$$-mg - kv_y = m \frac{dv_y}{dt}$$

$$-\frac{1}{m} = \frac{1}{mg + kv_y} \frac{dv_y}{dt}$$

$$\int -\frac{1}{m} dt = \int \frac{1}{mg + kv_y} dv_y$$

$$\int -\frac{1}{m} dt = \frac{1}{k} \int \frac{k}{mg + kv_y} dv_y$$

Using the transformation: $\int \frac{f'(x)}{f(x)} dx = \ln(f(x))$

$$-\frac{1}{m} t = \frac{1}{k} \ln(mg + kv_y) + c$$

Substitute: at $t = 0$, $v_y = u \sin \theta$

$$0 = \frac{1}{k} \ln(mg + k u \sin \theta) + c$$

$$c = -\frac{1}{k} \ln(mg + k u \sin \theta)$$

Substitute c into previous equation:

$$-\frac{1}{m}t = \frac{1}{k} \ln(mg + kv_y) - \frac{1}{k} \ln(mg + ku \sin \theta)$$

$$-\frac{k}{m}t = \ln\left(\frac{mg + kv_y}{mg + ku \sin \theta}\right)$$

$$e^{-\frac{k}{m}t} = \frac{mg + kv_y}{mg + ku \sin \theta}$$

$$(mg + ku \sin \theta)e^{-\frac{k}{m}t} = mg + kv_y$$

$$v_y = \frac{(mg + ku \sin \theta)e^{-\frac{k}{m}t} - mg}{k}$$

$$v_y = \left(\frac{m}{k}g + u \sin \theta\right)e^{-\frac{k}{m}t} - \frac{m}{k}g$$

Deriving the vertical displacement equation

Let the vertical displacement be represented by the y coordinate.

$$y = \frac{dv_y}{dt}$$

$$y = \int \left(\frac{m}{k}g + u \sin \theta\right)e^{-\frac{k}{m}t} - \frac{m}{k}g dt$$

$$y = \left(\frac{m}{k}g + u \sin \theta\right) \int e^{-\frac{k}{m}t} - \int \frac{m}{k}g dt$$

$$y = \left(-\frac{m}{k}\right)\left(\frac{m}{k}g + u \sin \theta\right)e^{-\frac{k}{m}t} - \frac{m}{k}gt + c$$

Substitute: at $t = 0, y = 0$

$$0 = \left(-\frac{m}{k}\right)\left(\frac{m}{k}g + u \sin \theta\right) + c$$

$$c = \left(\frac{m}{k}\right)\left(\frac{m}{k}g + u \sin \theta\right)$$

Substitute c into previous equation:

$$y = \left(-\frac{m}{k}\right)\left(\frac{m}{k}g + u \sin \theta\right)e^{-\frac{k}{m}t} - \frac{m}{k}gt + \left(\frac{m}{k}\right)\left(\frac{m}{k}g + u \sin \theta\right)$$

$$y = \left(\frac{m}{k}\right)\left(\frac{m}{k}g + u \sin \theta\right) - \left(\frac{m}{k}\right)\left(\frac{m}{k}g + u \sin \theta\right)e^{-\frac{k}{m}t} - \frac{m}{k}gt$$

$$y = \frac{m}{k}\left(\frac{m}{k}g + u \sin \theta\right)(1 - e^{-\frac{k}{m}t}) - \frac{m}{k}gt$$

Summary

$$v_x = u \cos \theta e^{-\frac{k}{m}t}$$

$$v_y = \left(\frac{m}{k}g + u \sin \theta\right)e^{-\frac{k}{m}t} - \frac{m}{k}g$$

$$x = \frac{m}{k}u \cos \theta (1 - e^{-\frac{k}{m}t})$$

$$y = \frac{m}{k}\left(\frac{m}{k}g + u \sin \theta\right)(1 - e^{-\frac{k}{m}t}) - \frac{m}{k}gt$$

g is the gravitational force and would thus vary depending on the altitude or planet the object is thrown on.

k must make up all other factors that affect the size of the force of air resistance acting on the object. Therefore it would depend on the size and shape of the object as well as the density of the medium it is traveling through (Eg. Air or water).

Comparison

Modeling the difference between a vacuum and air resistance

By substituting values into the equations that have been derived, the difference in the shape of the path traveled can be examined.

Let: $m=1\text{ kg}$ $u=50\text{ ms}^{-1}$ $k=0.2\text{ kg s}^{-1}$ $g=-9.81\text{ ms}^{-2}$

*These values are used for the remainder of this investigation unless specified differently.

Using the vacuum displacement equations:

$$x = u \cos \theta t$$

$$y = \frac{1}{2} g t^2 + u \sin \theta t$$

Using the air resistance inclusive displacement equations:

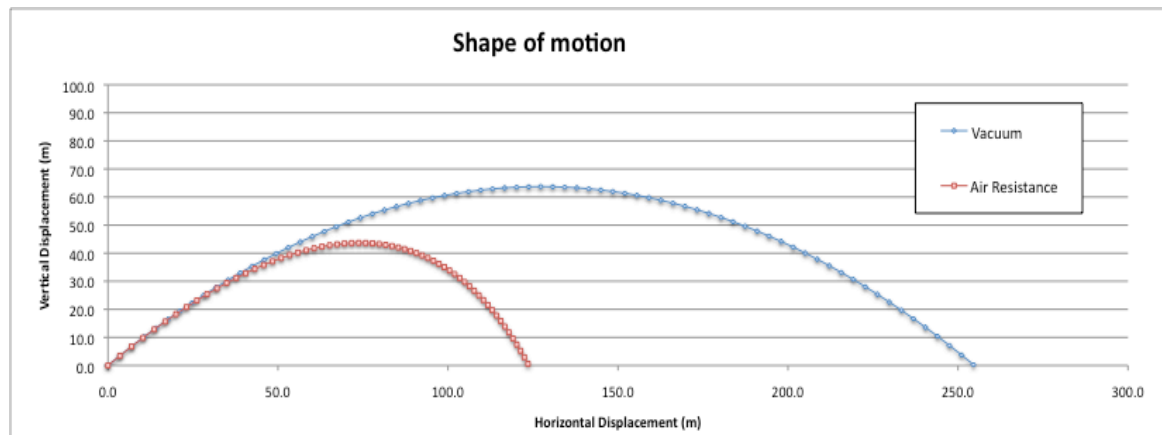
$$x = \frac{m}{k} u \cos \theta (1 - e^{-\frac{k}{m} t})$$

$$y = \frac{m}{k} \left(\frac{m}{k} g + u \sin \theta \right) (1 - e^{-\frac{k}{m} t}) - \frac{m}{k} g t$$

To the right is a sample of the table that the equations produce with defined values.

The values in the table produce the graph below.

Time	Vacuum		Air Resistance	
	x	y	x	y
0.0	0.0	0.0	0.0	0.0
0.1	2.6	4.2	2.6	4.2
0.2	5.3	8.3	5.2	8.2
0.3	7.9	12.3	7.8	12.1
0.4	10.5	16.2	10.3	15.9
0.5	13.1	20.0	12.8	19.5
0.6	15.8	23.8	15.3	23.0
0.7	18.4	27.4	17.8	26.4
0.8	21.0	30.9	20.2	29.7
0.9	23.6	34.3	22.6	32.8
1.0	26.3	37.6	25.0	35.7
1.1	28.9	40.9	27.4	38.6
1.2	31.5	44.0	29.7	41.3
1.3	34.1	47.0	32.0	43.9
1.4	36.8	49.9	34.3	46.4
1.5	39.4	52.8	36.6	48.8
1.6	42.0	55.5	38.8	51.0
1.7	44.7	58.2	41.1	53.1
1.8	47.3	60.7	43.3	55.1
1.9	49.9	63.1	45.5	57.0
2.0	52.5	65.5	47.6	58.7



The graph shows how when air resistance is taken into account, the path of the object is no longer a parabola. With air resistance, the final horizontal displacement as well as the maximum height the object reaches is decreased. For this example, the horizontal distance that the object would travel is approximately 130m shorter. Therefore, air resistance must be taken into account when making predictions on the distance that an object will fly when it is thrown. This is what I expected as air resistance results in a force that is opposite to the motion so it would diminish the distance that the object would travel with the same initial conditions.

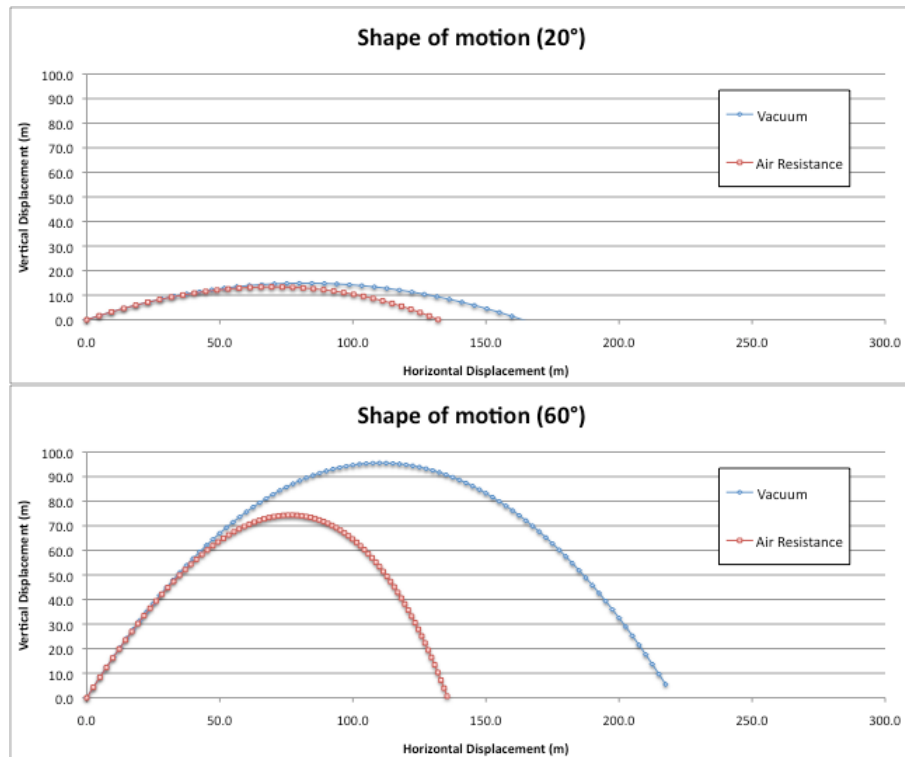
Changing the values of the variables

I created an interactive excel sheet that when I input different values for mass, initial velocity, the initial angle, the constant k, and gravitational force, the horizontal and vertical displacements are changed, and automatically graphed.

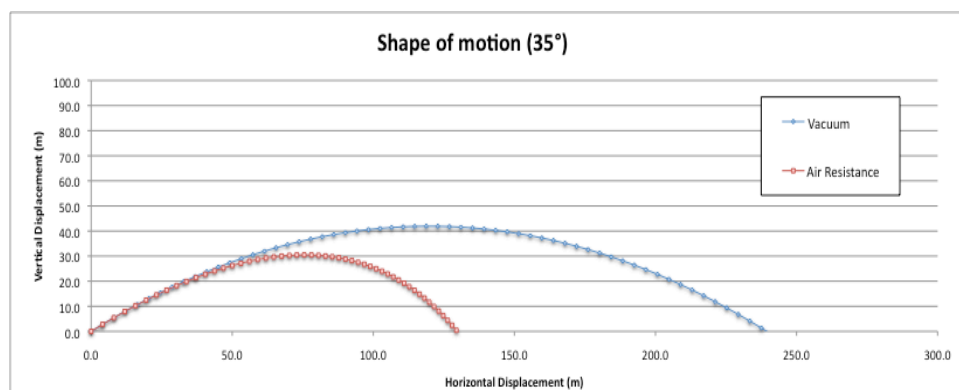
	A	B	C	D	E
1	m=1kg	1			
2	u=50m/s	50			
3	angle=45	0.78539816			
4	k=1kg/s	0.2			
5	t change (s)	0.1			
6	g	-9.81			
7		Vacuum		Air Resistance	
8	Time	x	y	x	y
9	0.0	0.0	0.0	0.0	0.0
10	0.1	3.5	3.5	3.5	3.5
11	0.2	7.1	6.9	6.9	6.7
12	0.3	10.6	10.2	10.3	9.9
13	0.4	14.1	13.4	13.6	12.8
14	0.5	17.7	16.5	16.8	15.6
15	0.6	21.2	19.4	20.0	18.3
16	0.7	24.7	22.3	23.1	20.8
17	0.8	28.3	25.1	26.1	23.2
18	0.9	31.8	27.8	29.1	25.4
19	1.0	35.4	30.5	32.0	27.5
20	1.1	38.9	33.0	34.9	29.4
21	1.2	42.4	35.4	37.7	31.2
22	1.3	46.0	37.7	40.5	32.9
23	1.4	49.5	39.9	43.2	34.4
24	1.5	53.0	42.0	45.8	35.8
25	1.6	56.6	44.0	48.4	37.1
26	1.7	60.1	45.9	51.0	38.3
27	1.8	63.6	47.7	53.4	39.3

Initial angle

Below are some examples of the graphs that were created when the angle was changed but all other factors remained constant. As what would be expected, the maximum height reached increases as the initial angle increases.



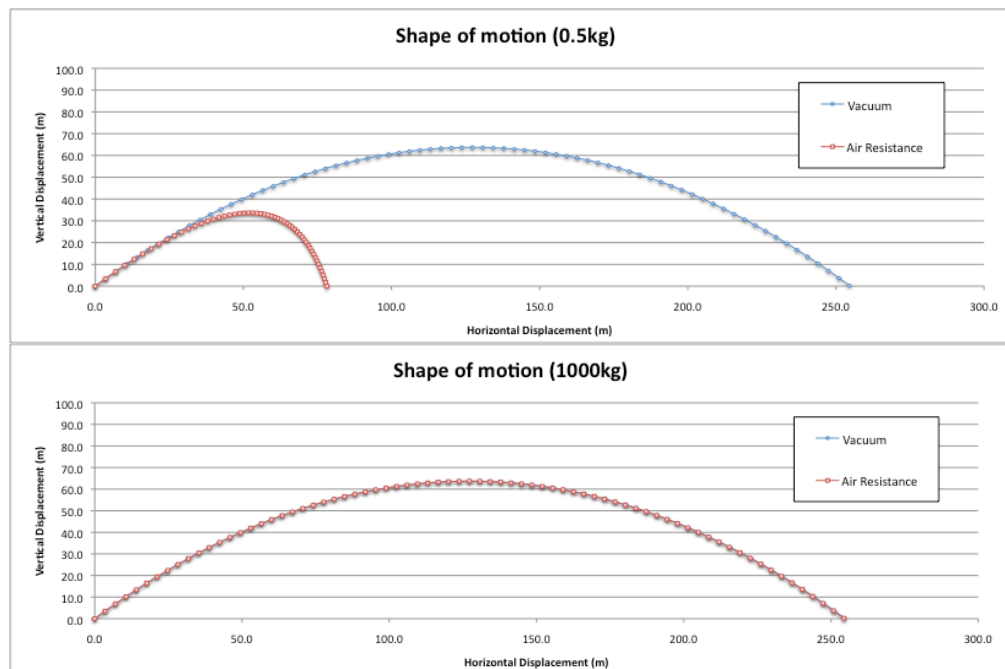
After multiple trials, I found that when air resistance is taken into account, the horizontal displacement was greatest when the initial angle was equal to 35° (compared to 45° when air resistance is negligible).



Mass

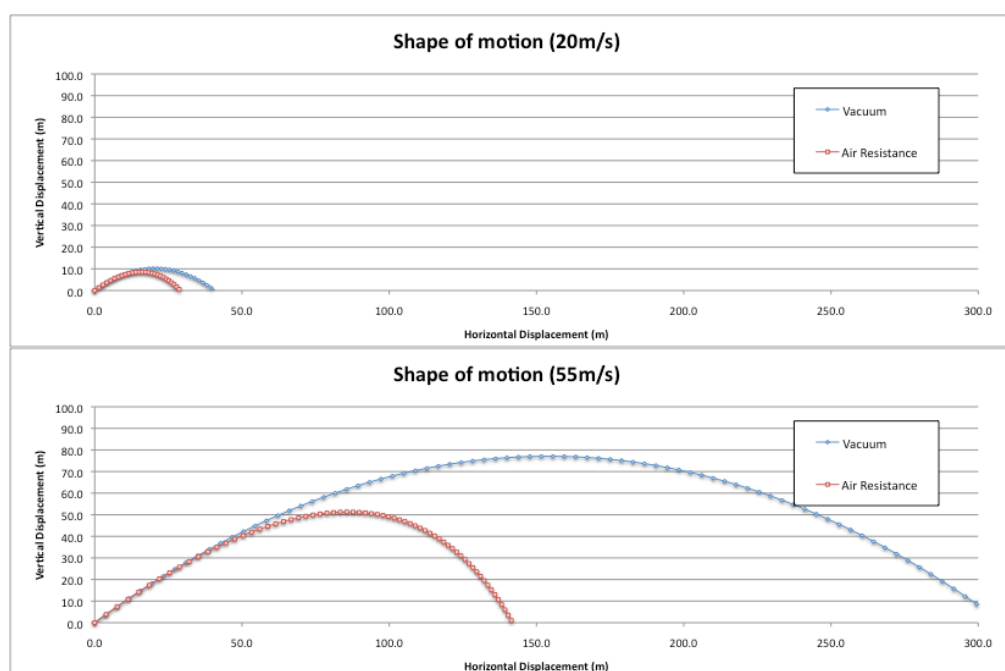
As the mass of the object increases, the motion of the object with air resistance approaches the motion of the object in a vacuum. The motion in a vacuum is

shown to be independent of mass. This is because mass is not a variable in the projectile motion equations for a vacuum.



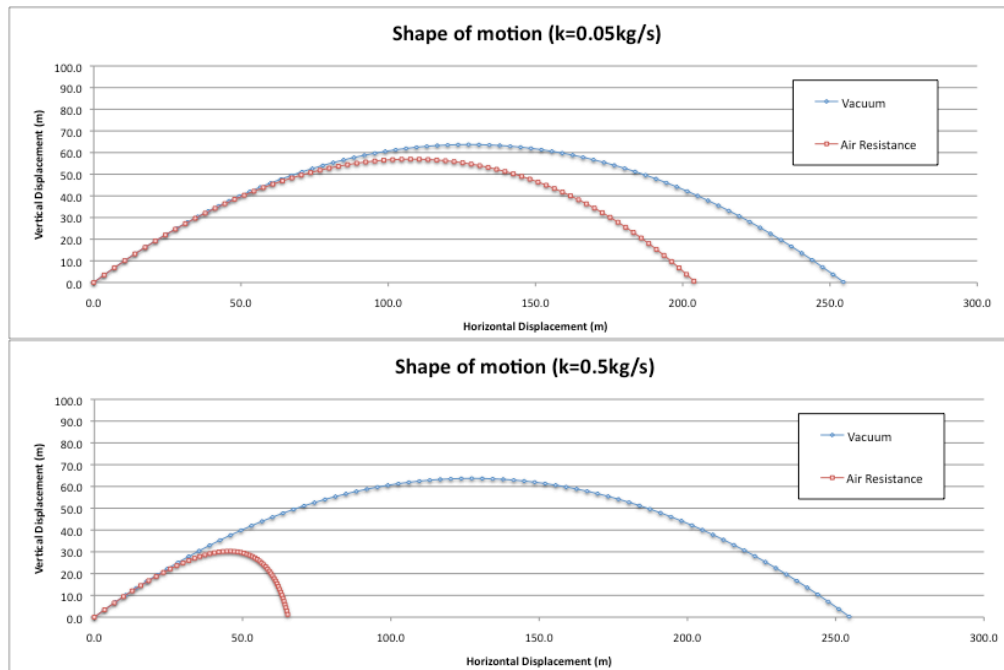
Initial velocity

The initial velocity is affecting both the projectiles in a vacuum and with air resistance. As the magnitude of the initial velocity increases, both the maximum height reached and the final horizontal displacement increases. It is also interesting to note that the smaller the initial velocity, the closer the motion of the object with air resistance is to the motion of the object in a vacuum. I assume that this stems from the fact that the magnitude of the force of air resistance depends on kv , and a smaller initial velocity would result in a smaller value for v .



The constant k

As the constant k increases, the magnitude of air resistance (kv) increases and therefore the motion with air resistance moves further away from the motion in a vacuum. Therefore, the constant k has a negative relationship to the final horizontal displacement. As k is a variable that describes the air resistance, it does not affect the motion of a projectile in a vacuum where air resistance does not exist. This also supports that k is representing the factors that determine the strength of air resistance (Eg. Shape/size of object and density of medium).



Works Cited

The Physics Classroom. (2015). *Types of Forces*. Retrieved February 13, 2015, from The Physics Classroom: <http://www.physicsclassroom.com/class/newtlaws/Lesson-2/Types-of-Forces>

The Physics Classroom. (2015). *What is a projectile?* Retrieved February 16, 2015, from the Physics Classroom: <http://www.physicsclassroom.com/class/vectors/Lesson-2/What-is-a-Projectile>