Mathematics SL Internal Assessment

Does my dog walk more than me?

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Introduction

Last year, my pet dachshund had to undergo surgery to correct his hernia. The probability of whether he would be able to walk again depended on how well his rehabilitation progressed. The final stages of rehabilitation involved rebuilding his leg muscle by taking him on short walks. I interpreted "short" as a 15-minute walk once around the block. However, I was later advised by the vet to shorten this distance. The vet claimed that dogs can walk up to double the distance that we do because they often walk circles around us. The aim of this investigation is to examine the validity of the vet's claim by creating a scenario in which I can graph the equation of my dog's walk and my own. In doing so, I can explore the relationship between the distance that dogs walk in comparison to their owners.

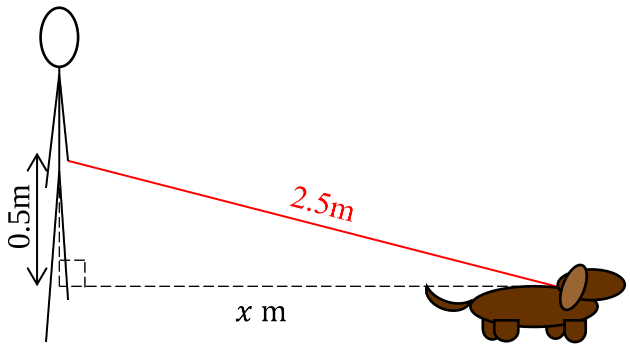
Scenario 1

When developing this scenario, I recognized that I would not be able to find an equation for my dog or my own path without establishing rules. Hence, the first challenge I faced was determining what would be an appropriate value for each rule. Since I want my investigation to be authentic, I heavily prioritized realism to base my choices on. Moreover, to ensure my process will be easily understood, I chose more simple numbers for my speeds. Hence, based on these factors, I decided to implement the following rules in my investigation:

1. A 2.5m leash is extended to its maximum length throughout the walk.
2. I walk at a constant speed of 1m/s.
3. My dog walks at a constant speed of 2m/s relative to a stationary point.

Step 1: Finding the distance between my dog and myself

The following diagram models the distance between my dog and myself given that his leash is extended to maximum length (2.5m):



From this, we can apply Pythagoras Theorem to find x:

$$a^2 + b^2 = c^2$$
$$0.5^2 + x^2 = 2.5^2$$
$$x = \sqrt{6}$$

Step 2: Finding an equation for my walk

Since my dog's range of motion is restricted by his leash, his path revolves around me as the central point. Hence, step 2 of this investigation is to graph an equation for my own walk. Because the variable of time is involved, I chose to use parametric equations to model this scenario rather than a rectangular equation (an equation composed of variables x and y which can be graphed on a regular Cartesian plane). A curve in the plane is parameterized if the set of coordinates, (x, y), on the curve are represented as functions of a variable t.

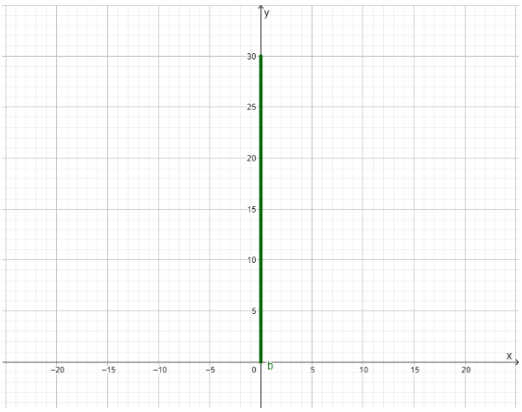
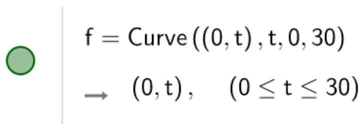
$$x = f(t)$$
$$y = g(t)$$

These functions describe the x and y position of an object moving in a plane. The advantages of using parametric equations include being able to graph curves that are not functions and being able to represent a third variable. The disadvantage when sketching parametric curves is that we must pick a set range for t based on intuition. Hence, there is always the possibility that the entire curve is not modeled. Moreover, since parametric equations are not part of the syllabus, this approach would require that I learn the basics of the topic myself.

The third variable in this scenario, called a parameter, is time (t) in seconds. The purpose of this investigation is to compare the distance that dogs walk to that of their owners **over a set amount of time**. I chose to represent time between $0 \leq t \leq 30$ for this investigation. This range ensures enough distance is covered for comparison and prevented the model from being overly complicated.

Given that I am walking in a straight line towards my destination, we could graph my walk as a rectangular equation where $x = \mathbb{R}$; where $\mathbb{R}$ represents the set of all real numbers (real numbers are values that can be represented as a quantity along a number line). We use parametric equations instead because there is an advantage of being able to incorporate the variable of time (t) and demonstrate a direction of motion in the model. Let's say that the road is represented by $x = 0$. In this case, I would be walking in one direction along $x = 0$ at 1m/s. In other words, the y coordinate, representing meters, will increase proportionally to t. From this information, we can base the following parametric equations:

$$x = 0$$
$$y = t$$
$$0 \leq t \leq 30$$

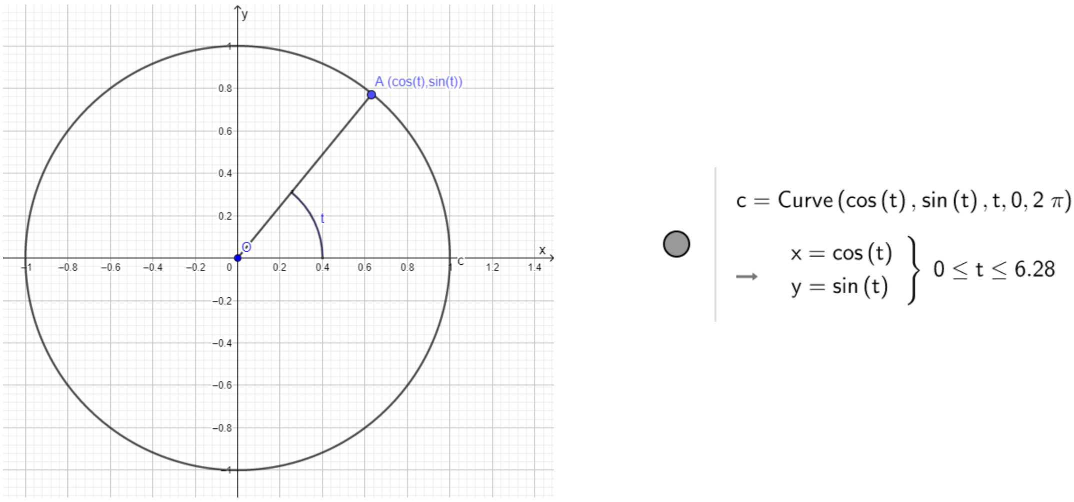



(Hohenwarter, Markus)

Step 3: Finding an equation for my dog's walk

For this scenario, I am walking at a speed of 1m/s along $x = 0$. My dog, who is always $\sqrt{6}$m from my coordinates, walks in circles around me at a speed of 2m/s. To derive an equation for my dog's walk, I chose to build off the parametric equations for a unit circle (a circle centered at $(0,0)$ with radius 1). This would allow me to build my equation based on what I already know about trigonometric identities. If $\mathrm{A}(x, y)$ moves around the unit circle such that OA makes an angle of t with a positive x-axis, then:

$$x = \cos t$$
$$y = \sin t$$



(Hohenwarter, Markus)

The parametric equations for a unit circle would represent the following situation: I am standing at the origin (0,0) and my dog is continuously walking in a circle, of radius 1m, around me. To satisfy the requirements of this investigation, the equations must consider the distance between my dog and I, as well as the speed that we are walking. We can base our changes using periodic function formulas, $y = A\sin(Bx + C) + D$ and $y = A\cos(Bx + C) + D$. The variables in this formula refer to amplitude (A), period ($\frac{2\pi}{b}$), horizontal shift (C), and vertical shift (D).

<u>Amplitude</u>

The first step is to incorporate the distance between my dog and myself into the equations. This has to do with the amplitude (A) of the graph (the amplitude refers to the distance from the midpoint to the highest or lowest point of the function). We found in step 1 that my dog is always $\sqrt{6}$m from my coordinates. The radius of the circle that my dog walks and therefore the amplitude of this graph is $\sqrt{6}$.

$$x = \sqrt{6}\cos t$$
$$y = \sqrt{6}\sin t$$

<u>Period</u>

The next step is to find and incorporate the period (the time required for the function to make one full oscillation) into the equations. For this scenario, an oscillation is one full revolution walked around me by my dog.

Distance required to make one full oscillation (where radius is $\sqrt{6}$):

$$\text{Circumference} = 2\pi r$$
$$\text{Circumference} = 2\sqrt{6}\pi$$

Time required for my dog to make one full oscillation (where my dog walks at 2m/s):

$$d = s \times t$$
$$2\sqrt{6}\pi = 2 \times t$$
$$t = \sqrt{6}\pi$$

Hence, we can find b:

$$\text{Period} = \frac{2\pi}{b}$$
$$\sqrt{6}\pi = \frac{2\pi}{b}$$
$$b = \frac{2}{\sqrt{6}}$$

Then,

$$x = \sqrt{6}\cos\frac{2t}{\sqrt{6}}$$
$$y = \sqrt{6}\sin\frac{2t}{\sqrt{6}}$$

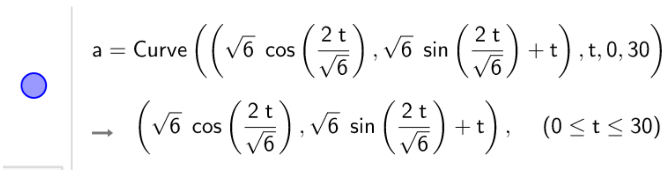
Vertical Shift

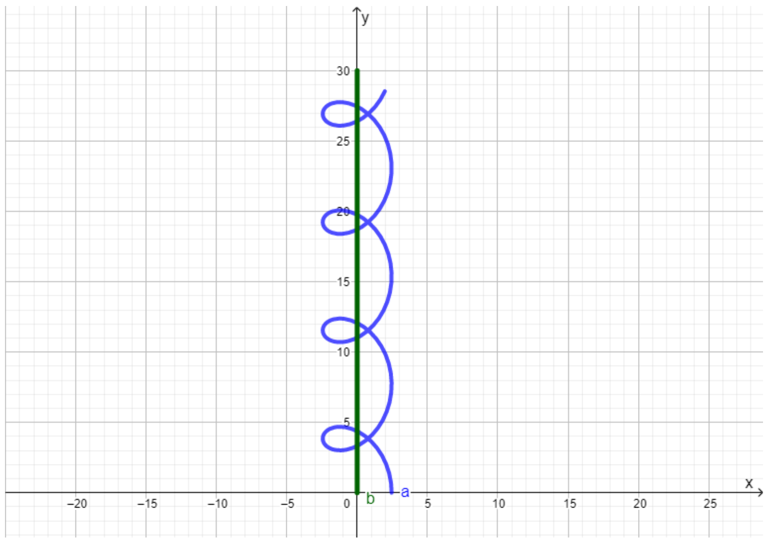
The final step is the addition of a vertical shift (D) to the parametric equation for y (a vertical shift is a measure of how far the graph has shifted vertically from its' initial position). Because my dog's path revolves around me as the central point, the equations for my walk must be implemented into his. This can be accomplished by adding the parametric equations from my path to his:

$$x = \sqrt{6}\cos\frac{2t}{\sqrt{6}} + 0$$
$$= \sqrt{6}\cos\frac{2t}{\sqrt{6}}$$
$$y = \sqrt{6}\sin\frac{2t}{\sqrt{6}} + t$$
$$0 \le t \le 30$$

By adding the variable t to the equation for y, we can sketch the effect time has on distance walked.

Graphing my dog's walk





Step 4: Calculating the distance (in meters) walked

Distance I walked

The parameter, time (t) in seconds, was represented between $0 \leq t \leq 30$ for this investigation. Given that I am walking 1m/s in a straight line to my destination, we can find the distance that I walked using the following equation:

$$speed = \frac{distance}{time}$$
$$1 = \frac{distance}{30}$$
$$distance = 30 \text{ meters}$$

Distance my dog walked

I had difficulty with this step of my investigation because in SL Math, we are not taught parametric equations in depth. Hence, I approached this challenge by researching online to see if there was an existing method to derive the length of a parametric curve. I found the following formula in a tutorial by Paul Dawkins from California State University Northridge:

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

(Dawkins, Paul)

We can use the formula above to determine the length of the parametric curve for $0 \leq t \leq 30$ given by the following parametric equations:

$$x = \sqrt{6}\cos\frac{2t}{\sqrt{6}}$$
$$y = \sqrt{6}\sin\frac{2t}{\sqrt{6}} + t$$

We will first need to find the derivatives of each equation. For my dog's x equation,

$$\frac{dx}{dt}\left(\sqrt{6}\cos\frac{2t}{\sqrt{6}}\right)$$
$$= \sqrt{6} \times \frac{dx}{dt}\left(\cos\frac{2t}{\sqrt{6}}\right)$$
$$= \sqrt{6}\left(-\sin\frac{2t}{\sqrt{6}}\right) \times \frac{dx}{dt}\left(\frac{2t}{\sqrt{6}}\right)$$
$$= -\sqrt{6} \times \frac{2t}{\sqrt{6}} \times \frac{dx}{dt}(t) \times \sin\frac{2t}{\sqrt{6}}$$
$$= -2\sin\frac{2t}{\sqrt{6}}$$

For my dog's y equation,

$$\frac{dy}{dt}\left(\sqrt{6}\sin\frac{2t}{\sqrt{6}} + t\right)$$
$$= \sqrt{6} \times \frac{dy}{dt}\left(\sin\frac{2t}{\sqrt{6}} + t\right)$$

$$= \sqrt{6}\cos\frac{2t}{\sqrt{6}} \times \frac{dy}{dt}\left(\frac{2t}{\sqrt{6}} + t\right)$$
$$= \sqrt{6}\left(\frac{2t}{\sqrt{6}} \times \frac{dy}{dt}(t) + \frac{dy}{dt}(t)\right)\cos\left(\frac{2t}{\sqrt{6}} + t\right)$$
$$= \left(\frac{2}{\sqrt{6}} + 1\right) \times \sqrt{6}\cos\left(\frac{2t}{\sqrt{6}} + t\right)$$
$$= \left(\sqrt{6} + 2\right) \times \cos\left(\frac{2t}{\sqrt{6}} + t\right)$$

The length is then,

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$
$$L = \int_{0}^{30} \sqrt{\left(-2\sin\frac{2t}{\sqrt{6}}\right)^2 + \left(\left(\sqrt{6} + 2\right) \times \cos\left(\frac{2t}{\sqrt{6}} + t\right)\right)^2}\, dt$$

Since this integration would be extremely difficult to do by hand, I chose to use the integrate function on my graphing display calculator (GDC) to solve for L:

$$L = 97.78 \text{ meters}$$

<u>Scenario 2</u>

When I take my dog for a walk, realistically it is never just a straight line to my destination. Let's say we walk towards an intersection and another dog (dog B) crosses in front of us. In this scenario, my dog (dog A) would stop circling me and begin to pursue dog B. The resulting curve is referred to as a pursuit curve, where dog A is always directed towards dog B. For this investigation, dog A and B will move with uniform velocities (1m/s). To avoid unnecessary complication, if my dog begins to curve, we will assume I follow his path without resistance. This scenario considers a common obstacle in dog walks and therefore presents a more realistic conclusion to the aim outlined in the introduction.

Step 1: Finding coordinates for dog A

Same as scenario 1, I am walking dog A from the origin $(0,0)$ along $x = 0$. Since dog B will initially be walking perpendicular to dog A, we can represent dog B's walk as $y = \mathbb{R}$; where $\mathbb{R}$ represents the set of all real numbers. For simplicity, let's say dog B is walking at 1m/s along $y = 3$. We can show this information with the following vector equation:

$$\vec{B} = \begin{pmatrix} t \\ 3 \end{pmatrix}$$

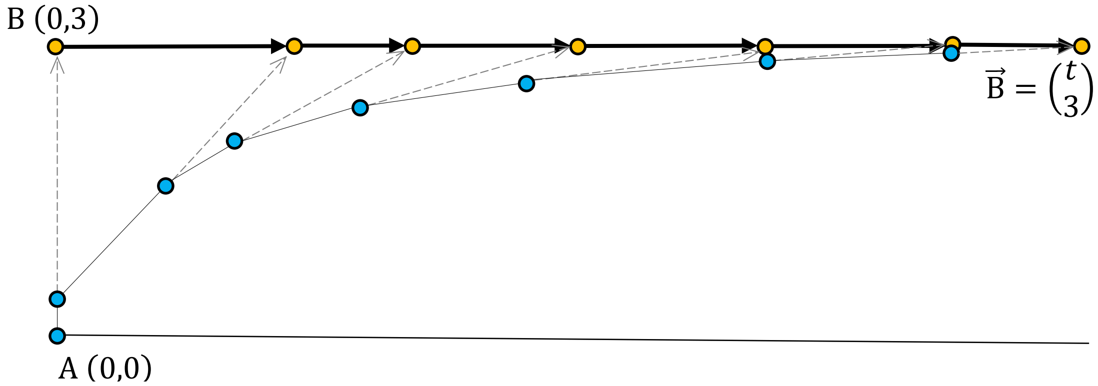
My initial approach to model dog A was to find a standard formula for equations of pursuit. Research online directed me to a pursuit formula, where I substituted $\vec{A} = \begin{pmatrix} x \\ y \end{pmatrix}$ and $\vec{B} = \begin{pmatrix} t \\ 3 \end{pmatrix}$.

$$\frac{\left(\vec{B} - \vec{A}\right) \times \dot{\vec{A}}}{\left|\vec{B} - \vec{A}\right|} = 1$$

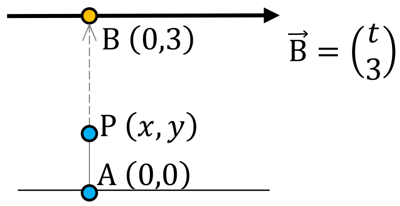
$$1 = \frac{\begin{pmatrix} t - x \\ 3 - y \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}}{\begin{vmatrix} t - x \\ 3 - y \end{vmatrix}}$$

(Weisstein, Eric W.)

Unfortunately, I cannot solve the derived equation because of the "overdot" (indicating differentiation in respect to time). Instead of using a formula to find the pursuit curve, I changed my approach and decided to use vectors to find coordinates for dog A. Coordinates will be found for small time intervals of constant velocity: I chose to use 0.1 second time intervals. The two dogs will be in eye range when dog A is at $(0,0)$ and dog B is at $(0,3)$. From this position onwards, dog A will always move in the direction of dog B:



First, we must find dog A's position when $t = 0.1$ (represented by P in the diagram below).



Before we can find P (x, y), we must find $\overrightarrow{AB}$ (direction vector) and $\widehat{AB}$ (unit vector).

Finding $\overrightarrow{AB}$:

$$\overrightarrow{AB} = \vec{B} - \vec{A}$$
$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$
$$\overrightarrow{AB} = \begin{pmatrix} 0 \\ 3 \end{pmatrix}$$

Finding $\widehat{AB}$:

$$\widehat{AB} = \frac{\overrightarrow{AB}}{|\overrightarrow{AB}|}$$
$$\widehat{AB} = \frac{\begin{pmatrix} 0 \\ 3 \end{pmatrix}}{\sqrt{0^2 + 3^2}}$$
$$\widehat{AB} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Coordinate $\mathrm{P}(x, y)$ can thus be found by,

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{\mathrm{A}} + t\widehat{\mathrm{AB}}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \end{pmatrix}$$
$$x = 0$$
$$\begin{aligned} y &= t \\ &= 0.1 \end{aligned}$$

Therefore, after 0.1 seconds, dog A has moved to $\mathrm{P}(0, 0.1)$. Since calculating each coordinate by hand would be inefficient, I chose to set up a model on Excel.

Step 2: Setting up an Excel model to find dog A coordinates

	A	B	C	D	E	F	G	H
1			DOG A		DOG B		DIRECTION VECTOR	
2		TIME (t)	X	Y	X	Y	X	Y
3		0.0	0.00	0.00	0.00	3.00	0.00	3.00
4		0.1	0.00	0.10	0.10	3.00	0.10	2.90
5		0.2	0.00	0.20	0.20	3.00	0.20	2.80
6		0.3	0.01	0.30	0.30	3.00	0.29	2.70
7		0.4	0.02	0.40	0.40	3.00	0.38	2.60
8		0.5	0.04	0.50	0.50	3.00	0.46	2.50
9		0.6	0.05	0.60	0.60	3.00	0.55	2.40
10		0.7	0.08	0.69	0.70	3.00	0.62	2.31
11		0.8	0.10	0.79	0.80	3.00	0.70	2.21
12		0.9	0.13	0.89	0.90	3.00	0.77	2.11
13		1.0	0.17	0.98	1.00	3.00	0.83	2.02

Time

As you go down a row, the value for time will increase by 0.1 intervals. This column represents the overall time which has passed.

Direction Vector ($\overrightarrow{\mathrm{AB}}$)

	DOG A		DOG B		DIRECTION VECTOR	
TIME (t)	X	Y	X	Y	X	Y
0.0	0.00	0.00	0.00	3.00	0.00	3.00
0.1	0.00	0.10	0.10	3.00	=E4-C4	2.90

To find any direction vector,

$$\overrightarrow{\mathrm{AB}} = \vec{\mathrm{B}} - \vec{\mathrm{A}}$$
$$\overrightarrow{\mathrm{AB}} = \begin{pmatrix} \mathrm{B}_x \\ \mathrm{B}_y \end{pmatrix} - \begin{pmatrix} \mathrm{A}_x \\ \mathrm{A}_y \end{pmatrix}$$
$$\mathrm{AB}_x = \mathrm{B}_x - \mathrm{A}_x$$
$$\mathrm{AB}_y = \mathrm{B}_y - \mathrm{A}_y$$

We can use the information stored in excel cells to calculate the direction vector $\overrightarrow{\mathrm{AB}}$. When $t = 0.1$, we use cells C4 (stores A_x), D4 (stores A_y), E4 (stores B_x), and F4 (stores B_y) as following:

$$\begin{aligned} \mathrm{AB}_x &= \mathrm{B}_x - \mathrm{A}_x \\ &= \mathrm{E4} - \mathrm{C4} \end{aligned}$$
$$\begin{aligned} \mathrm{AB}_y &= \mathrm{B}_y - \mathrm{A}_y \\ &= \mathrm{F4} - \mathrm{D4} \end{aligned}$$

Dog B

Given that dog B is walking along $y = 3$, we can deduce that his y-coordinate will not change with time. Since he walks at a speed of 1m/s, his x-coordinate will increase proportionally to time: each 0.1 second increase, dog B will move an additional 0.1m from the previous x-coordinate.

$$s = \frac{d}{t}$$
$$1 = \frac{d}{0.1}$$
$$d = 0.1$$

Dog A

$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{\mathrm{A}} + t\widehat{\mathrm{AB}}$$
$$\begin{pmatrix} x \\ y \end{pmatrix} = \vec{\mathrm{A}} + t\frac{\overrightarrow{\mathrm{AB}}}{|\overrightarrow{\mathrm{AB}}|}$$

(Note: For this equation, t is always substituted by 0.1 regardless of how much time has passed overall. This is because we are finding the coordinates for $\mathrm{P}(x, y)$ with different direction vectors each calculation. If at any other point in these report t is referred to, it should be defined as the overall time which has passed.)

Hence,

$$x = \vec{\mathrm{A}}_x + 0.1\frac{\overrightarrow{\mathrm{AB}}_x}{|\overrightarrow{\mathrm{AB}}|}$$

$$y = \vec{\mathrm{A}}_y + 0.1\frac{\overrightarrow{\mathrm{AB}}_y}{|\overrightarrow{\mathrm{AB}}|}$$

As an example, consider finding the y-coordinate for dog A when $t = 0.2$. To find the dog's new position, we must know the current y-coordinate for dog A (stored in cell D4), and the corresponding direction vector $\overrightarrow{\mathrm{AB}}$ (stored in cells G4 and H4).

SUM =D4+((H4*0.1)/SQRT(G4^2+H4^2))

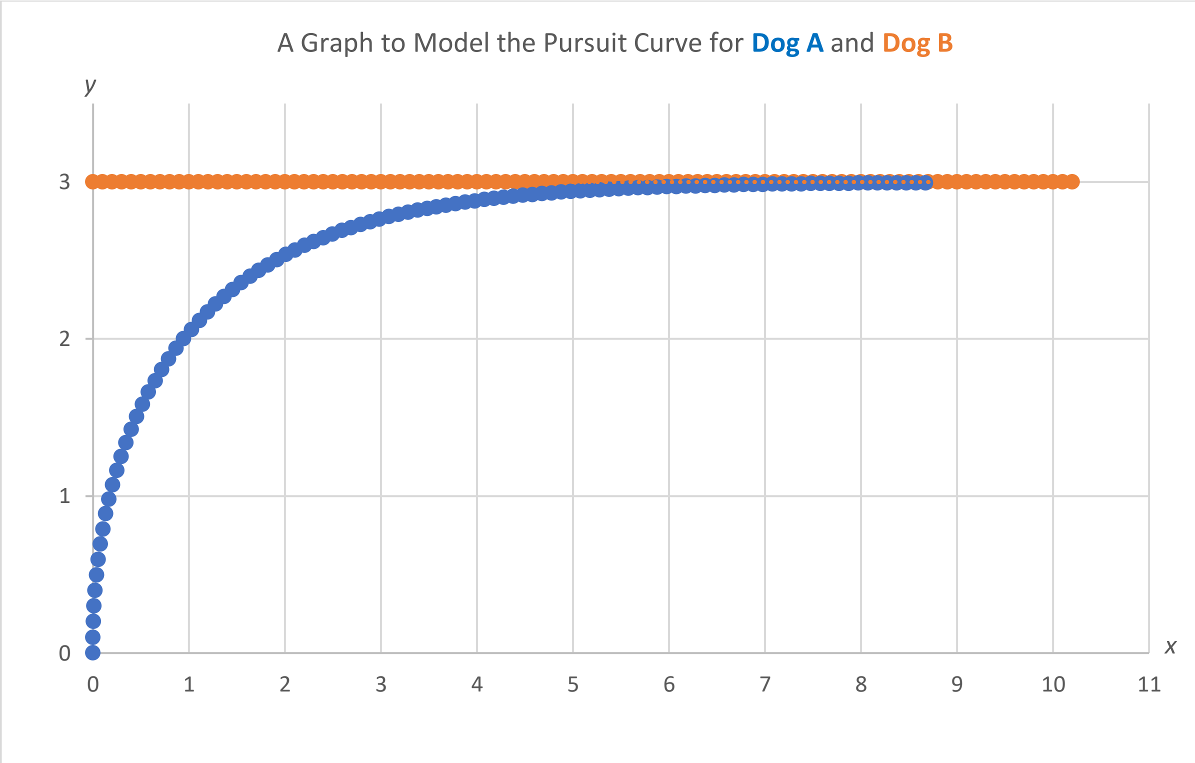
	A	B	C	D	E	F	G	H
1			DOG A		DOG B		DIRECTION VECTOR	
2		TIME (t)	X	Y	X	Y	X	Y
3		0.0	0.00	0.00	0.00	3.00	0.00	3.00
4		0.1	0.00	0.10	0.10	3.00	0.10	2.90
5		0.2	0.00	=D4+((H4*0.1)/S	0.20	3.00	0.20	2.80

$$y = \vec{\mathrm{A}}_y + 0.1\frac{\overrightarrow{\mathrm{AB}}_y}{|\overrightarrow{\mathrm{AB}}|}$$

$$y = \mathrm{D4} + 0.1\frac{\mathrm{H4}}{\sqrt{\mathrm{G4}^2 + \mathrm{H4}^2}}$$

The same process is repeated for the x-coordinate. We can use the fill handle to copy the formula into other cells and then proceed to graph these coordinates.

Step 3: Graphing the Excel model for dog A and dog B



After graphing dog A's path, I recognized that the intervals between points were not consistent as they should have been. While this may become a source of error, I chose to continue as is after confirming my formulas were correct. While I had originally planned to rely on excel to find a line of best fit for the pursuit curve, I discovered that this would not be possible given the complexity of dog A's path. Instead, I will need to find a function which has similar characteristics to the pursuit curve graphed in Fig. 11.

Step 4: Finding and manipulating a function

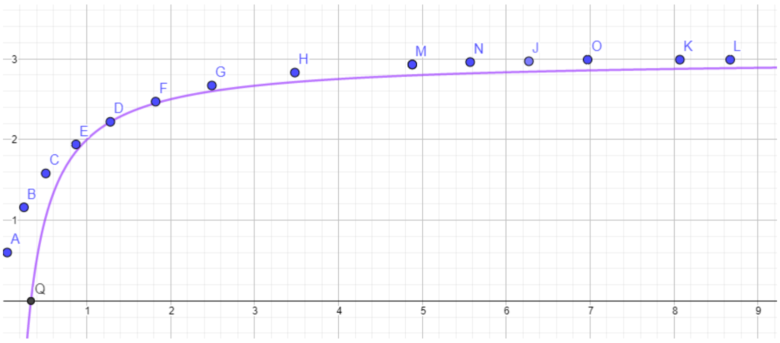
Finding a function

My approach to this task reminded me of how I found my dog's equation in scenario 1. Hence, like I did for that task, I will find a similar function to manipulate. Here, I noticed that $y = \tan^{-1} x$ and $y = -\frac{1}{x}$ have similar visual shapes. The defining feature of my pursuit curve is that it has asymptotes: the curve is bound by $0 \leq y \leq 3$ and $x \geq 0$. Therefore, I chose to manipulate $y = -\frac{1}{x}$ because I know this function has two asymptotes: $y = 0$ and $x = 0$. (Note: I will only consider the positive side of the x-axis from this point onwards.)

Manipulating a function

Instead of the asymptote being $y = 0$, I will vertically shift the equation by 3 so that the asymptote is $y = 3$. While the asymptote should really be $y = 3.\dot{0}1$, this is not possible to input on GeoGebra. Since this will not result in any statistically significant problems, I decided to leave it as 3. This equation will be graphed with 15 randomly chosen coordinates from the excel spreadsheet to show how it relates to the ideal pursuit curve.

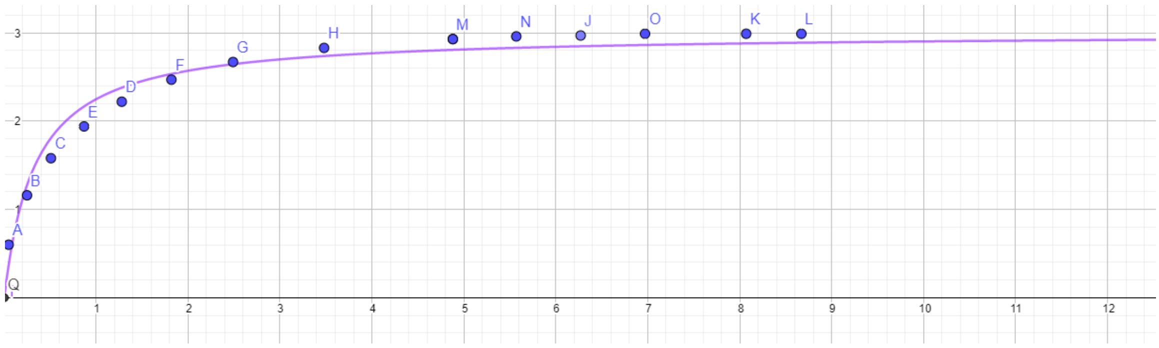
$$y = -\frac{1}{x} + 3$$



(Hohenwarter, Markus)

Point Q, where the curve intersects the x-axis, is $(0.33,0)$. For the curve to intersect the origin, we must horizontally shift it right by 0.33. The resulting curve and its' graph is,

$$y = -\frac{1}{x + 0.33} + 3$$



(Hohenwarter, Markus)

Conclusion

In the modelled 30 seconds for scenario 1, I walked 30 meters. In the same duration, my dog walked 97.78 meters. Hence, under the variables given for this scenario, we can deduce that my dog walked 3.26 times the distance I did. Therefore, the vet's claim is only partially valid because dogs walk triple, instead of double, the distance we do: a ratio that results from dogs circling their owners during walks. This discovery is of significance because it means owners are misunderstanding the intensity of exercise that their dogs are doing. If your pet is hurt, intense exercise can cause additional stress to the injury and hinder the rehabilitation progress. Therefore, it becomes very important for owners to recognize how much more their dog could be walking and adjust based on this recognition. The major limitation to this scenario is that it is heavily based on assumptions. If you've ever walked a dog, you would know that they don't continuously circle you. Therefore, in scenario 2, I investigated how dog A's walk changes based on the presence of dog B. This made me aware of how obstacles in a walk can decrease the dog's total walking distance. Instead of circling me, my dog would change his path and walk along a pursuit curve: given by $y = -\frac{1}{x+0.33} + 3$. This suggests that the obstacles result in the total distance being walked decreasing. Since we did not consider obstacles when finding that my dog walks 3.26 times the distance I do, we can assume that this is unrealistic. Considering how obstacles decrease the total walking distance, it becomes more legitimate for the vet to claim a dog walks double the distance I do. However, it should be noted that the extent to which a dog's paths is affected by obstacles, whether that be other dogs or fire hydrants, is dependent on the dog. Moreover, the extent to which a dog circles their own is also subjective. These assumptions and generalizations are the biggest limitations to this investigation. Additionally, the inability to find an exact equation for the pursuit curve (having to rely on approximation) also becomes a source of error. The investigation could have been improved by combining what was found in both scenarios to determine what distance my dog and I respectively would have walked if we encountered a certain number of dogs. This would have allowed for a more realistic conclusion to be drawn and therefore should be considered as a possible extension. My methodology was a strength considering how I adapted my process to find a pragmatic solution to problems: such as using excel after my initial pursuit curve formula was too complex to calculate. This investigation taught me how to use digital skills to solve real life application problems: for instance, using a variety of softwares to draw a conclusion for scenario 2. The investigation could be improved greatly if I could evaluate distance walked based on different types of dogs. A possible extension would therefore be to find the distance walked for three different situations: where dog A circles his owner without obstacle, dog B pursuits another dog, and dog C walks towards each fire hydrant.

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Appendix

	A	B	C	D	E	F	G	H
1			DOG A		DOG B		DIRECTION VECTOR	
2		TIME (t)	X	Y	X	Y	X	Y
3		0.0	0.00	0.00	0.00	3.00	0.00	3.00
4		0.1	0.00	0.10	0.10	3.00	0.10	2.90
5		0.2	0.00	0.20	0.20	3.00	0.20	2.80
6		0.3	0.01	0.30	0.30	3.00	0.29	2.70
7		0.4	0.02	0.40	0.40	3.00	0.38	2.60
8		0.5	0.04	0.50	0.50	3.00	0.46	2.50
9		0.6	0.05	0.60	0.60	3.00	0.55	2.40
10		0.7	0.08	0.69	0.70	3.00	0.62	2.31
11		0.8	0.10	0.79	0.80	3.00	0.70	2.21
12		0.9	0.13	0.89	0.90	3.00	0.77	2.11
13		1.0	0.17	0.98	1.00	3.00	0.83	2.02
14		1.1	0.20	1.07	1.10	3.00	0.90	1.93
15		1.2	0.25	1.16	1.20	3.00	0.95	1.84
16		1.3	0.29	1.25	1.30	3.00	1.01	1.75
17		1.4	0.34	1.34	1.40	3.00	1.06	1.66
18		1.5	0.40	1.42	1.50	3.00	1.10	1.58
19		1.6	0.45	1.50	1.60	3.00	1.15	1.50
20		1.7	0.51	1.58	1.70	3.00	1.19	1.42
21		1.8	0.58	1.66	1.80	3.00	1.22	1.34
22		1.9	0.65	1.73	1.90	3.00	1.25	1.27
23		2.0	0.72	1.81	2.00	3.00	1.28	1.19
24		2.1	0.79	1.87	2.10	3.00	1.31	1.13
25		2.2	0.87	1.94	2.20	3.00	1.33	1.06
26		2.3	0.94	2.00	2.30	3.00	1.36	1.00
27		2.4	1.02	2.06	2.40	3.00	1.38	0.94
28		2.5	1.11	2.12	2.50	3.00	1.39	0.88
29		2.6	1.19	2.17	2.60	3.00	1.41	0.83
30		2.7	1.28	2.22	2.70	3.00	1.42	0.78
31		2.8	1.37	2.27	2.80	3.00	1.43	0.73
32		2.9	1.45	2.31	2.90	3.00	1.45	0.69
33		3.0	1.54	2.36	3.00	3.00	1.46	0.64
34		3.1	1.64	2.40	3.10	3.00	1.46	0.60
35		3.2	1.73	2.44	3.20	3.00	1.47	0.56
36		3.3	1.82	2.47	3.30	3.00	1.48	0.53
37		3.4	1.92	2.51	3.40	3.00	1.48	0.49
38		3.5	2.01	2.54	3.50	3.00	1.49	0.46
39		3.6	2.11	2.57	3.60	3.00	1.49	0.43
40		3.7	2.20	2.59	3.70	3.00	1.50	0.41
41		3.8	2.30	2.62	3.80	3.00	1.50	0.38
42		3.9	2.40	2.65	3.90	3.00	1.50	0.35
43		4.0	2.49	2.67	4.00	3.00	1.51	0.33
44		4.1	2.59	2.69	4.10	3.00	1.51	0.31
45		4.2	2.69	2.71	4.20	3.00	1.51	0.29
46		4.3	2.79	2.73	4.30	3.00	1.51	0.27
47		4.4	2.89	2.75	4.40	3.00	1.51	0.25
48		4.5	2.98	2.76	4.50	3.00	1.52	0.24
49		4.6	3.08	2.78	4.60	3.00	1.52	0.22
50		4.7	3.18	2.79	4.70	3.00	1.52	0.21
51		4.8	3.28	2.81	4.80	3.00	1.52	0.19
52		4.9	3.38	2.82	4.90	3.00	1.52	0.18
53		5.0	3.48	2.83	5.00	3.00	1.52	0.17
54		5.1	3.58	2.84	5.10	3.00	1.52	0.16
55		5.2	3.68	2.85	5.20	3.00	1.52	0.15
56		5.3	3.78	2.86	5.30	3.00	1.52	0.14
57		5.4	3.88	2.87	5.40	3.00	1.52	0.13
58		5.5	3.98	2.88	5.50	3.00	1.52	0.12
59		5.6	4.08	2.89	5.60	3.00	1.52	0.11
60		5.7	4.18	2.89	5.70	3.00	1.52	0.11
61		5.8	4.28	2.90	5.80	3.00	1.52	0.10
62		5.9	4.38	2.91	5.90	3.00	1.52	0.09
63		6.0	4.48	2.91	6.00	3.00	1.52	0.09
64		6.1	4.58	2.92	6.10	3.00	1.52	0.08
65		6.2	4.68	2.92	6.20	3.00	1.52	0.08
66		6.3	4.78	2.93	6.30	3.00	1.52	0.07
67		6.4	4.88	2.93	6.40	3.00	1.52	0.07
68		6.5	4.98	2.94	6.50	3.00	1.52	0.06
69		6.6	5.08	2.94	6.60	3.00	1.52	0.06
70		6.7	5.18	2.95	6.70	3.00	1.52	0.05
71		6.8	5.28	2.95	6.80	3.00	1.52	0.05
72		6.9	5.38	2.95	6.90	3.00	1.52	0.05
73		7.0	5.47	2.96	7.00	3.00	1.53	0.04
74		7.1	5.57	2.96	7.10	3.00	1.53	0.04
75		7.2	5.67	2.96	7.20	3.00	1.53	0.04
76		7.3	5.77	2.96	7.30	3.00	1.53	0.04
77		7.4	5.87	2.97	7.40	3.00	1.53	0.03
78		7.5	5.97	2.97	7.50	3.00	1.53	0.03
79		7.6	6.07	2.97	7.60	3.00	1.53	0.03
80		7.7	6.17	2.97	7.70	3.00	1.53	0.03
81		7.8	6.27	2.97	7.80	3.00	1.53	0.03
82		7.9	6.37	2.98	7.90	3.00	1.53	0.02
83		8.0	6.47	2.98	8.00	3.00	1.53	0.02
84		8.1	6.57	2.98	8.10	3.00	1.53	0.02
85		8.2	6.67	2.98	8.20	3.00	1.53	0.02
86		8.3	6.77	2.98	8.30	3.00	1.53	0.02
87		8.4	6.87	2.98	8.40	3.00	1.53	0.02
88		8.5	6.97	2.98	8.50	3.00	1.53	0.02
89		8.6	7.07	2.99	8.60	3.00	1.53	0.01
90		8.7	7.17	2.99	8.70	3.00	1.53	0.01
91		8.8	7.27	2.99	8.80	3.00	1.53	0.01
92		8.9	7.37	2.99	8.90	3.00	1.53	0.01
93		9.0	7.47	2.99	9.00	3.00	1.53	0.01
94		9.1	7.57	2.99	9.10	3.00	1.53	0.01
95		9.2	7.67	2.99	9.20	3.00	1.53	0.01
96		9.3	7.77	2.99	9.30	3.00	1.53	0.01
97		9.4	7.87	2.99	9.40	3.00	1.53	0.01
98		9.5	7.97	2.99	9.50	3.00	1.53	0.01
99		9.6	8.07	2.99	9.60	3.00	1.53	0.01
100		9.7	8.17	2.99	9.70	3.00	1.53	0.01
101		9.8	8.27	2.99	9.80	3.00	1.53	0.01
102		9.9	8.37	2.99	9.90	3.00	1.53	0.01
103		10.0	8.47	2.99	10.00	3.00	1.53	0.01
104		10.1	8.57	2.99	10.10	3.00	1.53	0.01
105		10.2	8.67	3.00	10.20	3.00	1.53	0.00