# Do Dogs Know Calculus?

IB Mathematics Higher Level Internal Assessment


Name: [redacted]

School: Osaka International School

Candidate Number: [redacted]


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## Introduction

In 2003, a professor named Tim Pennings went to Lake Michigan with his dog Elvis to play fetch. He noticed that Elvis was very excited to fetch the ball in the water and inferred that Elvis wanted to reach the ball in the shortest amount of time. He then thought that there could be three different ways for Elvis to get to the ball in the shortest amount of time, first by reducing the distance to the ball, second by reducing the swimming time (since Elvis's swimming speed was much slower than his running speed), and third by running across the beach and then at some point plunging into the water and swimming the rest of the way. Elvis fetched the ball using the third option, but that wasn't it. From doing calculations with calculus, Pennings found that Elvis almost always plunged into the water at a certain point, which minimized the time taken for that route the most! (Pennings)

## Aim

In Part 1 of this investigation, I will be investigating the same situation that Pennings experimented with his dog, and calculate if his dog Elvis's route really takes the shortest time by doing calculations using my own example situation. In Parts 2 and 3 I will be investigating new situations that Pennings did not experiment with his dog to see if in these new situations, applying Elvis's route will still give us the shortest time to get to the ball.

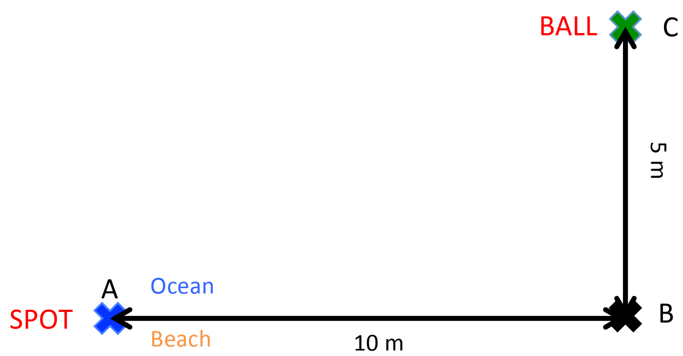
## Rationale

I chose this topic because it was interesting to me. When I first read the problem I couldn't believe that Elvis would actually know the fastest way to get to the ball so I wanted to investigate and calculate using my own example situation if applying Elvis's route would give me the shortest time to ball. I also wanted to expand on the topic and try different situations to see if there are certain limits to when we can apply Elvis's route to get the fastest route to the ball. Also, I was interested in applying calculus to a real life situation and seeing other possible calculus applications to real life from this investigation.

## Part 1: Standing at the edge of the water

*<u>Description of the situation-</u>*

In this situation, our dog, Spot, is standing at the edge of the water at point A, like with Pennings's experiment. The ball is at point C, 5 meters from the shore at point B, and from A to B the distance is 10 meters. Spot's swimming speed is $k$ m/sec and his running speed is $7k$ m/sec, where we are using the same speeds as Pennings's dog Elvis (Cavers).



*Situation 1: Reducing the distance to the ball-*

One way to reduce the time to get to the ball would be by using the shortest route to the ball. The shortest route would be from point A to point C. This distance can be found by using the Pythagorean theorem:

$$\begin{aligned} a^2 + b^2 &= c^2 \\ (AB)^2 + (BC)^2 &= (AC)^2 \\ 10^2 + 5^2 &= (AC)^2 \\ AC &= 5\sqrt{5}\ \mathrm{m} \end{aligned}$$

This route involves Spot swimming the entire way. Therefore the total time it would take for the dog to use this route would be:

$$T(x) = \frac{5\sqrt{5}}{k}\ \mathrm{sec}$$

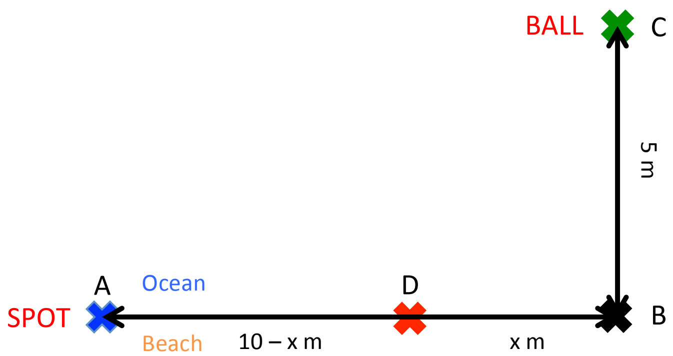
*Situation 2: Reducing swimming time-*

Considering the fact that Spot's running speed is seven times as fast as his walking speed, another way to try to reduce the time would be to minimize the distance swam. This route involves running across the beach from point A to point B, and then swimming from point B to point C. The total time taken would then be:

$$\begin{aligned} T(x) &= \frac{Distance\ AB}{Running\ Speed} + \frac{Distance\ BC}{Swimming\ Speed} \\ &= \frac{10}{7k} + \frac{5}{k} \\ &= \frac{45}{7k}\ \mathrm{sec} \end{aligned}$$

*Situation 3: Using Elvis's route-*

From Pennings's findings, he found that his dog ran across the beach to a certain point, and then swam the rest of the way to the ball. So where is this 'certain point' in our hypothetical situation? In the diagram, this 'certain point' is represented by point D.



Let us assume that Spot runs from point A to point D across the beach, and then from the D to C he swims. We want to find out the distance Spot runs across to point D to minimize the overall time to get to the ball.

To find the distance that Spot runs across the beach, we need to find the minimum point of the function of the time it takes for Spot to reach the ball.

$$Time\ taken\ from\ point\ A\ to\ D = \frac{Distance\ AD}{Running\ Speed} = \frac{10-x}{7k}$$

$$Time\ taken\ from\ point\ D\ to\ C = \frac{Distance\ DC}{Swimming\ Speed} = \frac{\sqrt{x^2+25}}{k}$$

Therefore we can write the function of time as follows:

$$T(x) = \frac{10-x}{7k} + \frac{\sqrt{x^2+25}}{k}$$

The minimum point of the function $T(x)$ occurs when $T'(x) = 0$ therefore we need to find the derivative of the function $T(x)$:

$$T(x) = \frac{10-x}{7k} + \frac{\sqrt{x^2+25}}{k}$$

$$T'(x) = -\frac{1}{7k} + \frac{1}{k}\times\frac{1}{2}(x^2+25)^{-\frac{1}{2}}\times 2x$$

$$= -\frac{1}{7k} + \frac{x}{k\sqrt{x^2+25}}$$

[redacted]

When we set $T'(x) = 0$:

$$-\frac{1}{7k} + \frac{x}{k\sqrt{x^2+25}} = 0$$

$$\frac{x}{k\sqrt{x^2+25}} = \frac{1}{7k}$$

$$7kx = k\sqrt{x^2+25}$$

$$7x = \sqrt{x^2+25}$$

$$49x^2 = x^2 + 25$$

$$48x^2 = 25$$

$$x^2 = \frac{48}{25}$$

$$x = 1.39 \text{ m } (3\ s.f.)$$

To test if this value of $x$ is the minimum of the function $T(x)$, we do the second derivative test:

$$T''(x) = 0 + \frac{1}{k}\left\{\left(\frac{\left[1\cdot(x^2+25)^{-\frac{1}{2}}\right] - \left[x\cdot -\frac{1}{2}(x^2+25)^{-\frac{3}{2}}\cdot 2x\right]}{x^2+25}\right)\right\}$$

$$= \frac{1}{k}\times\frac{(2x^2+25)/(x^2+25)^{\frac{3}{2}}}{x^2+25}$$

$$= \frac{2x^2+25}{k(x^2+25)^{\frac{5}{2}}}$$

$$= \frac{2(1.39)+25}{k((1.39)^2+25)^{\frac{5}{2}}}$$

$$= Positive\ Number$$

Because the value of the second derivative is positive at $x = 1.39$, it is a minimum point of the function $T(x)$, telling us that when the dog runs across the beach for

$$10 - x \ = 10 - 1.39 = 8.61 \text{ m}$$

and then swims the rest of the way to the ball, this minimizes the time the most.

Using this route, the total time it would take would be:

$$T(x) = \frac{10 - x}{7k} + \frac{\sqrt{x^2 + 25}}{k}$$

$$= \frac{8.61}{7k} + \frac{\sqrt{1.39^2 + 25}}{k}$$

$$= \frac{44.94}{7k} \text{ sec}$$

*Comparison of the three situations-*

We can now compare the times of the three situations.
Situation 1:

$$T(x) = \frac{5\sqrt{5}}{k}$$

$$= \frac{11.2}{k} \text{sec} \ \ (3 \ s.f.)$$

Situation 2:

$$T(x) = \frac{45}{7k}$$

$$= \frac{6.43}{k} \text{sec} \ \ (3 \ s.f.)$$

Situation 3:

$$T(x) = \frac{44.94}{7k}$$

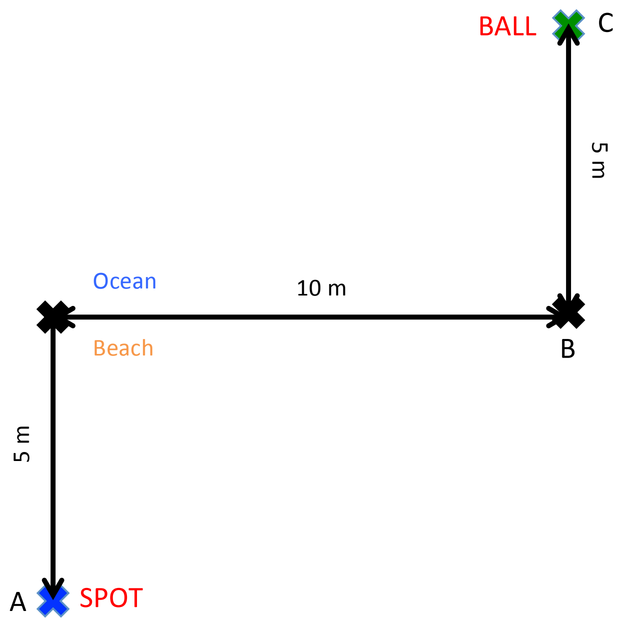
$$= \frac{6.42}{k} \text{sec} \ \ (3 \ s.f.)$$

Comparing the three situations, no matter what the dog's speed is, the time taken using the route in Situation 3 will always be the shortest time. Situation 1 probably takes the most time because Spot has to swim all the way to the ball and his swimming speed is significantly slower than his running speed. Additionally there may be little difference in the time taken for Situation 2 and 3 because the difference in the distance ran across the beach is less than 2 meters. So if the distance across the beach is longer, there is probably more benefit in using the route in Situation 3 to reduce time. Finally this shows that Elvis in Pennings's report did choose a route that minimized the time the most.

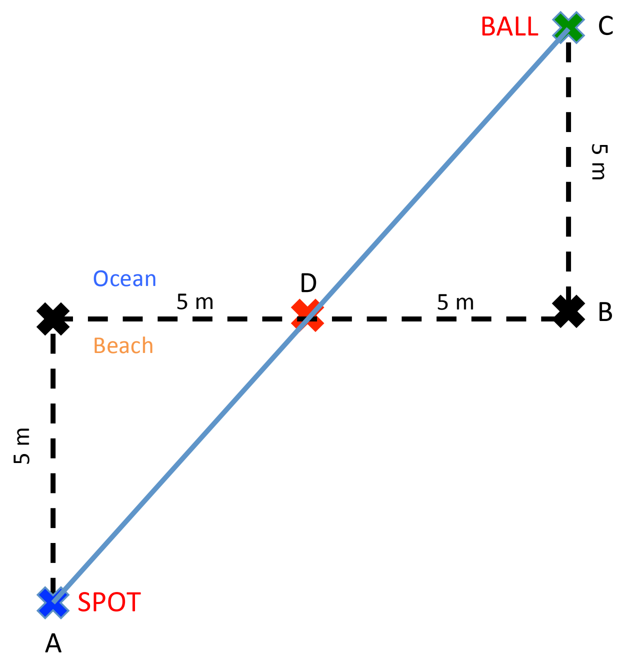
## Part 2: Standing at a distance from the water

*Description of the situation-*

In this situation, our dog, Spot, is standing at a distance from the water at point A. The ball is still at point C, which is 5 meters from the shore at point B, and the distance across the beach is 10 meters. Again Spot's swimming speed is $k$ m/sec and his running speed is $7k$ m/sec. We will apply the three situations in Part 1, reducing distance, reducing swimming time, and using Elvis's route, to Part 2 as well.




*Situation 1: Reducing the distance to the ball-*



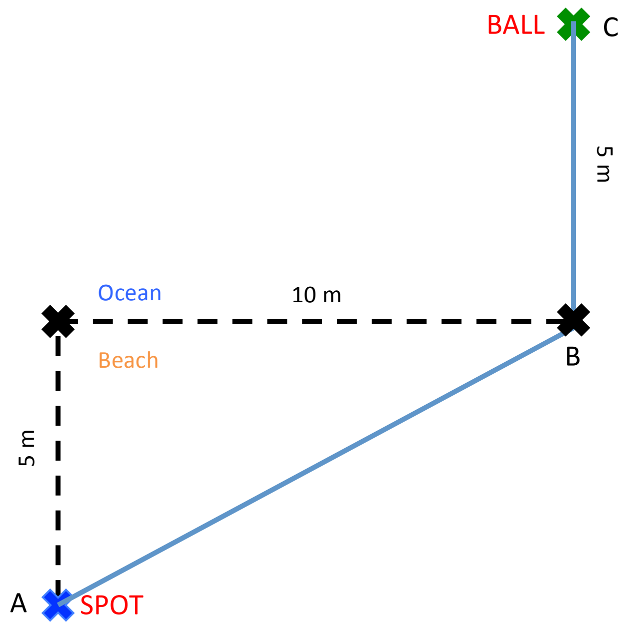
The route for the shortest distance to the ball in this situation would be going directly from point A to point C. This means running the distance from A to D, plunging into the water at D and then swimming from D to C. Therefore the time this route takes would be:

$$T(x) = \frac{Distance\ AD}{Running\ Speed} + \frac{Distance\ DC}{Swimming\ Speed}$$

$$= \frac{\sqrt{5^2 + 5^2}}{7k} + \frac{\sqrt{5^2 + 5^2}}{k}$$

$$= \frac{40\sqrt{2}}{7k}\ \text{sec}$$

*<u>Situation 2: Reducing swimming time-</u>*



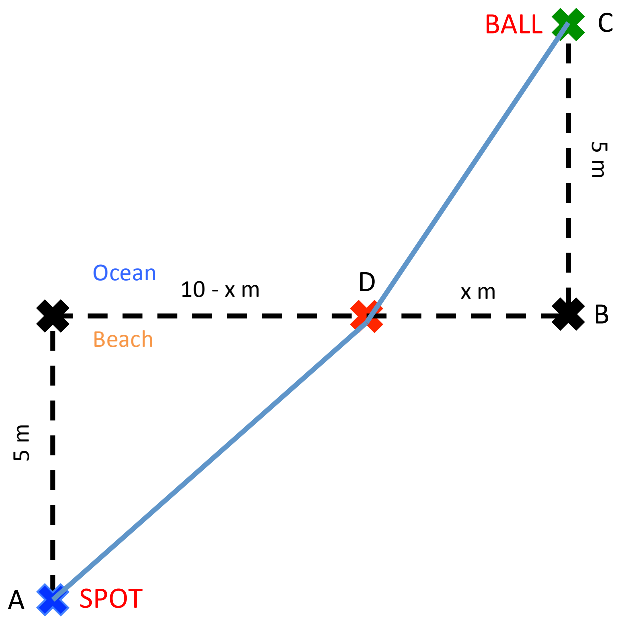

The route to reduce the swimming time would be to run from point A to B, and then swim from B to C. The total time taken for this route would be:

$$T(x) = \frac{Distance\ AB}{Running\ Speed} + \frac{Distance\ BC}{Swimming\ Speed}$$

$$= \frac{\sqrt{10^2 + 5^2}}{7k} + \frac{5}{k}$$

$$= \frac{35 + 5\sqrt{5}}{7k}\ \text{sec}$$

*Situation 3: Using Elvis' route-*




Using Elvis's route means that Spot will run from A to a certain point along the beach at point D, and then plunge into the water at D and then swim to C. The function of time can be written as:

$$T(x) = \frac{Distance\ AD}{Running\ Speed} + \frac{Distance\ DC}{Swimming\ Speed}$$

$$= \frac{\sqrt{5^2 + (10 - x)^2}}{7k} + \frac{\sqrt{x^2 + 5^2}}{k}$$

$$= \frac{\sqrt{x^2 - 20x + 125}}{7k} + \frac{\sqrt{x^2 + 25}}{k}$$

$$= \frac{\sqrt{x^2 - 20x + 125} + 7\sqrt{x^2 + 25}}{7k}$$

Again to minimize the time taken for this route, the minimum point of the function $T(x)$ occurs when $T'(x) = 0$ therefore we need to find the derivative of the function $T(x)$:

$$T'(x) = \frac{d}{dx}\left(\frac{\sqrt{x^2 - 20x + 125} + 7\sqrt{x^2 + 25}}{7k}\right)$$

$$= \frac{1}{7k}\frac{d}{dx}\left(\sqrt{x^2 - 20x + 125} + 7\sqrt{x^2 + 25}\right)$$

$$= \frac{1}{7k}\left(\frac{d}{dx}\left(\sqrt{x^2 - 20x + 125}\right) + \frac{d}{dx}\left(7\sqrt{x^2 + 25}\right)\right)$$

$$= \frac{1}{7k}\left(\left[\frac{1}{2}(x^2 - 20x + 125)^{-\frac{1}{2}} \cdot (2x - 20)\right] + 7\left[\frac{1}{2}(x^2 + 25)^{-\frac{1}{2}} \cdot (2x)\right]\right)$$

$$= \frac{1}{7k}\left(\frac{x - 10}{\sqrt{x^2 - 20x + 125}} + \frac{7x}{\sqrt{x^2 + 25}}\right)$$

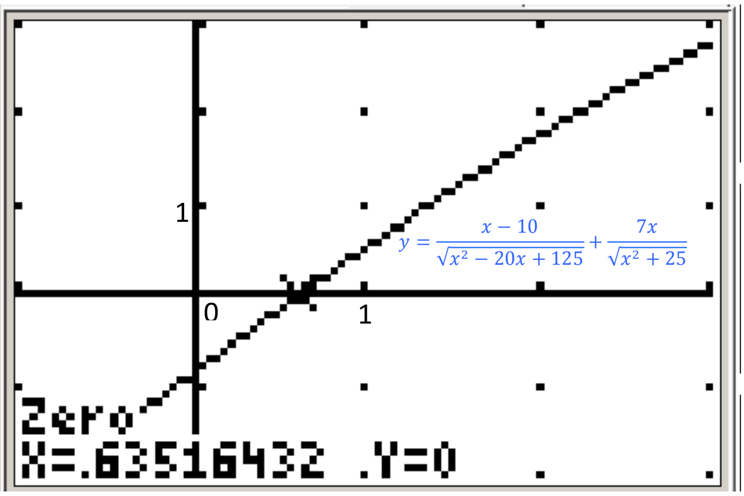
$$= \frac{\frac{x - 10}{\sqrt{x^2 - 20x + 125}} + \frac{7x}{\sqrt{x^2 + 25}}}{7k}$$

When we set $T'(x) = 0$:

$$\frac{\frac{x - 10}{\sqrt{x^2 - 20x + 125}} + \frac{7x}{\sqrt{x^2 + 25}}}{7k} = 0$$

$$\frac{x - 10}{\sqrt{x^2 - 20x + 125}} + \frac{7x}{\sqrt{x^2 + 25}} = 0$$

Using a graphing calculator, we can plot this graph and find where the graph equals 0, which will give us the value of x. This graph was made using TI-SmartView™ software.




$x = 0.635$ m $(3\ s.f.)$

Using this route, the total time it would take would be:

$$T(x) = \frac{\sqrt{x^2 - 20x + 125} + 7\sqrt{x^2 + 25}}{7k}$$

$$= \frac{\sqrt{0.635^2 - 20(0.635) + 125} + 7\sqrt{0.635^2 + 25}}{7k}$$

$$= \frac{45.90}{7k} \text{ sec}$$

*Comparison of the three situations-*

We can now compare the times of the three situations.

Situation 1:

$$T(x) = \frac{40\sqrt{2}}{7k}$$

$$= \frac{8.08}{k} \text{sec } (3\ s.f.)$$

Situation 2:

$$T(x) = \frac{35 + 5\sqrt{5}}{7k}$$

$$= \frac{6.60}{k} \text{sec } (3\ s.f.)$$

Situation 3:

$$T(x) = \frac{45.90}{7k}$$

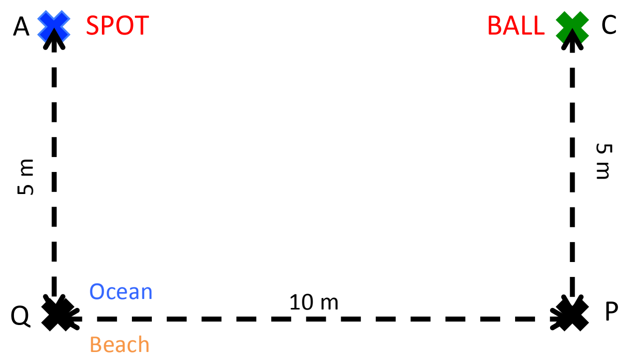
$$= \frac{6.56}{k} \text{sec } (3\ s.f.)$$

Comparing the three situations, no matter what the dog's speed is, the time taken using the route in Situation 3 will always be the shortest time. Situation 1 still takes the most time but there is less time difference between Situation 1 and Situations 2,3 because in Part 2 Spot has to run a longer distance compared to in Part 1. Again there is little difference in the time taken for Situation 2 and 3 because the difference in the distance ran across the beach is very small. So if the distance across the beach is longer, there is more benefit in using the route in Situation 3 to reduce time.

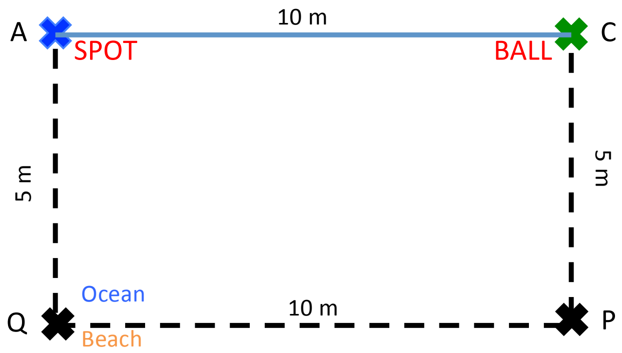
## Part 3: Standing in the water

*Description of the situation-*

In this situation, our dog, Spot, is standing in the water at point A which is 5 m from the shore from point Q. The ball is still at point C, which is 5 meters from the shore at point P, and the distance across the beach from Q to P is 10 meters. Again Spot's swimming speed is $k$ m/sec and his running speed is $7k$ m/sec. We will apply the three situations in Part 1 and 2, reducing distance, reducing swimming time, and using Elvis's route, to Part 3 as well.




*Situation 1: Reducing the distance to the ball/Reducing swimming time-*

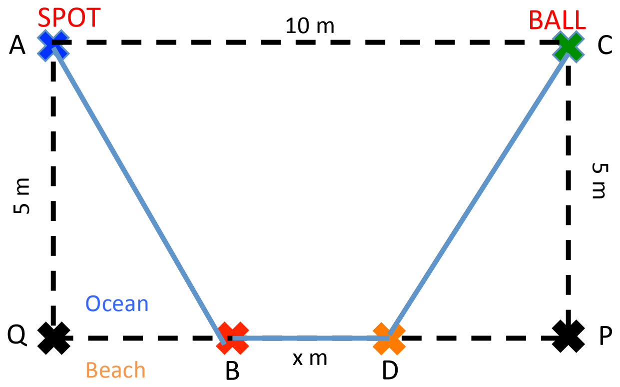



The shortest distance to the ball would be from point A to C, and this is also the route for reducing swimming time. The time taken for this route is:

$$T(x) = \frac{Distance\ AC}{Swimming\ Speed}$$

$$= \frac{10}{k}\ \text{sec}$$

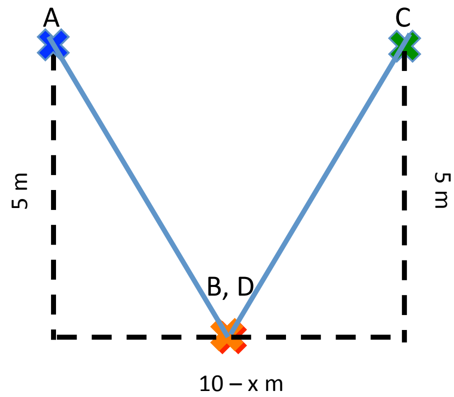
*Situation 2: Using a variation of Elvis's route-*



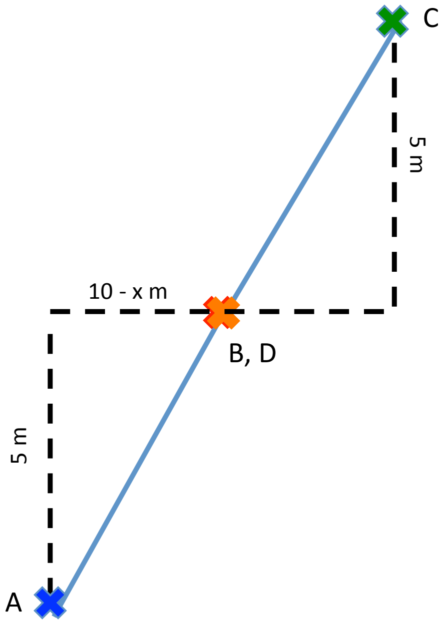

In this variation of Elvis's route, Spot is going to swim from A to B, run from B to D, and then swim D to C. The function of time taken can be written as:

$$T(x) = \frac{Distance\ AB + Distance\ CD}{Swimming\ Speed} + \frac{Distance\ BD}{Running\ Speed}$$

To find distance AB and distance CD, we can rearrange the diagram above by first putting points B and D on top of each other like this:



If we reflect the line AB over the 10 – x line, we can get the diagram below, which will enable us to find the distance AB + distance CD:




$$Distance\ AB + Distance\ CD = \sqrt{10^2 + (10 - x)^2}$$

Therefore the function of time can be written:

$$T(x) = \frac{Distance\ AB + Distance\ CD}{Swimming\ Speed} + \frac{Distance\ BD}{Running\ Speed}$$

$$= \frac{\sqrt{x^2 - 20x + 200}}{k} + \frac{x}{7k}$$

$$= \frac{7\sqrt{x^2 - 20x + 200} + x}{7k}$$

[redacted]

Again to minimize the time taken for this route, the minimum point of the function $T(x)$ occurs when $T'(x) = 0$ therefore we need to find the derivative of the function $T(x)$:

$$T'(x) = \frac{d}{dx}\frac{7\sqrt{x^2 - 20x + 200} + x}{7k}$$

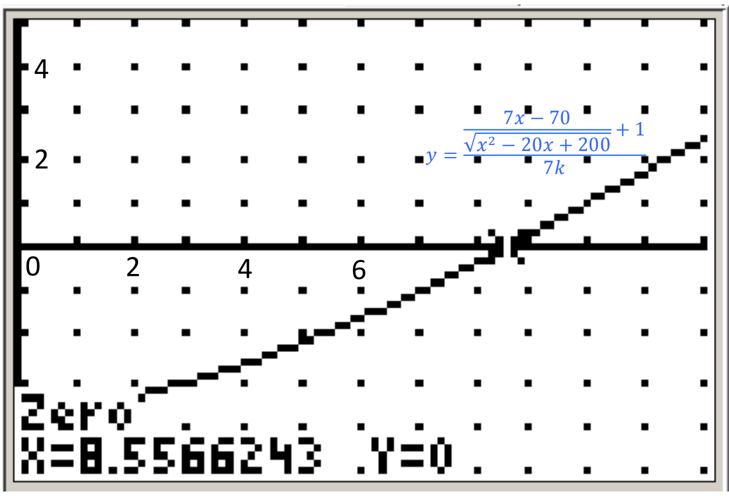
$$= \frac{1}{7k}\frac{d}{dx}7\sqrt{x^2 - 20x + 200} + x$$

$$= \frac{1}{7k}\left(\frac{d}{dx}7\sqrt{x^2 - 20x + 200} + \frac{d}{dx}x\right)$$

$$= \frac{1}{7k}\left(7\left(\frac{1}{2}(x^2 - 20x + 200)^{-\frac{1}{2}} \cdot (2x - 20)\right) + 1\right)$$

$$= \frac{\frac{7x - 70}{\sqrt{x^2 - 20x + 200}} + 1}{7k}$$

Using a graphing calculator, we can plot this graph and find where the graph equals 0, which will give us the value of x. This graph was made using TI-SmartView™ software.




$$x = 8.56 \text{ m } (3\ s.f.)$$

Using this route, the total time it would take would be:

$$T(x) = \frac{7\sqrt{8.56^2 - 20(8.56) + 200} + 8.56}{7k}$$

$$= \frac{79.28}{7k} \text{ sec}$$

*Comparison of the two situations-*

We can now compare the times of the two situations.

Situation 1:

$$T(x) = \frac{10}{k}\ \text{sec}$$

Situation 2:

$$T(x) = \frac{79.28}{7k}$$

$$= \frac{11.3}{k}\text{sec}\ \ (3\ s.f.)$$

Comparing the two situations, in this case using Elvis's route does not take the shortest time at all, and instead Situation 1 takes the least time no matter what the dog's speed is. This is probably because using Elvis's route for Situation 2 just involves too much swimming time, which slows down the total time. It may also be because there is a lot of distance that Spot has to cover. If Spot's location in the water was closer to the beach and/or his swimming speed was even slower and his running speed was even faster, perhaps using Elvis's route might take less time.

## Conclusion

In conclusion, for Part 1 and 2 of this investigation, using Elvis's route to get the ball appeared to take the shortest amount of time, and if we had used different distances it is most likely that Elvis's route would still yield the route that takes the shortest amount of time. However, in Part 3 we found that there are limits to this route as Elvis's route took the longest time if the dog was located in the water at the beginning. To investigate this problem further, we could try to figure out from what distance away from the shore the dog could be standing in the water, where using Elvis's route is still the most time efficient.

Using calculus to minimize the time taken for a route can be applied and be useful for other real life situations. For instance, in an emergency there is a need to escape as quick as possible. Therefore when people are mapping emergency maps, they may need to do calculations like I did in my investigation to map out the most time efficient escape route. There are many other applications where the technique of using calculus to minimize time can be used.

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