Investigating the Effectiveness of Penrose Tilings as a Model of Quasicrystals

How are aperiodic Penrose tilings similar to periodic tilings, and to what extent are they effective in modelling the structure of non-periodic quasicrystals?

Mathematics Extended Essay

Total Word Count: 3374

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Introduction: Tilings

A periodic tiling is one in which you can take a region of the tiling and tile the plane by translating (without rotating or reflecting) copies of that region.[1] These tilings can be made up of a single type or multiple types of regular polygons. Figure 1[2] shows some examples of these periodic tilings.

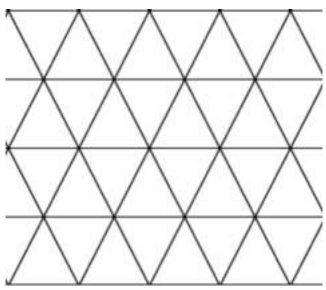


Figure 1.1: Periodic tiling with triangles

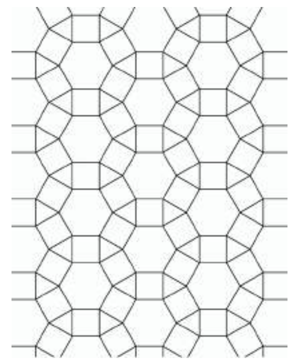


Figure 1.2: Periodic tiling with squares, triangles, and hexagons

In 1973, Roger Penrose, an English mathematician, discovered that there was a set of 6 tiles, known as pentacles, that force nonperiodicity[3], shown in Figure 2[4].

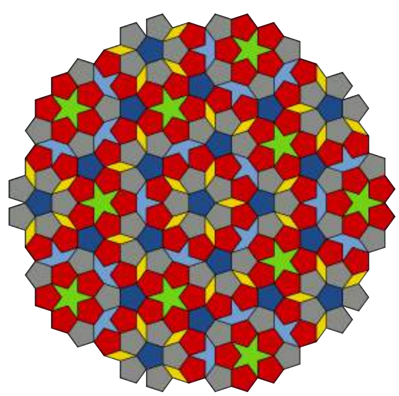
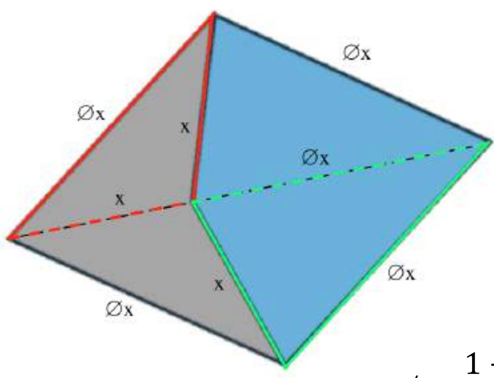


Figure 2: Penrose tiling made up of the 6 types of pentacles

[1] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers* (Washington: Mathematical Association of America, 1997), 1.
[2] Figure 1: Denis Zorin. The Archimedean tilings". 2003." Mass: Multiresolutional Adaptive Solid Subdivision. https://www.researchgate.net/figure/2564710_Figure-13-The-Archimedean-tilings-dual-to-Laves-Image-curtesy-of-Denis-Zorin
[3] David Austin. "Penrose Tiles Talk Across Miles." American Mathematical Society. Accessed January 08, 2018. http://www.ams.org/samplings/feature-column/fcarc-penrose.
[4] Figure 2: "A P1 tiling using Penrose's original set of six prototiles". 2009. Wikipedia. https://en.wikipedia.org/wiki/Penrose_tiling

Later, Penrose was able to reduce the number of tiles to two. There are two sets of these pair of tiles that can only tile nonperiodically: the dart and kite, and the thick and thin rhombs. In this essay, I will be focusing on the dart and kite tiles. The diagrams involving the dart and kite tiles used throughout this essay were created using a JavaScript program that I wrote (see Appendix A). Figure 3 displays the two tiles, which are derived from a rhombus; the gray tile shows the dart tile and the blue shows the kite tile. The ratio between the short side of the dart or kite tile to the long side is the golden ratio, $\frac{1+\sqrt{5}}{2}$. Additionally, the dart and kite tile can be split through the middle, producing 2 isosceles triangles respectively[5], outlined in red and green. If we split the rhombus into 4 isosceles triangles in this way, as shown in Figure 3, the angles of the tiles can be determined.



$$\emptyset = \frac{1+\sqrt{5}}{2}$$

Figure 3: Rhomb from which dart and kite tiles derived

Let us consider the two triangles using the cosine rule,

$$\cos\theta = \frac{b^2 + c^2 - a^2}{2bc}$$

First, for the red triangle (half the dart tile), let the long side be a, the two short sides be b and c, and θ be the angle between the two short sides.

$$\cos\theta = \frac{x^2 + x^2 - (\emptyset x)^2}{2 \cdot x \cdot x}$$

[5] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers* (Washington: Mathematical Association of America, 1997), 6.

$$\cos\theta = \frac{2x^2 - \emptyset^2 x^2}{2x^2}$$

$$\cos\theta = \frac{2 - \emptyset^2}{2}$$

$$\cos\theta = \frac{2 - \left(\frac{1+\sqrt{5}}{2}\right)^2}{2}$$

$$\theta = \cos^{-1}\frac{2 - \left(\frac{1+\sqrt{5}}{2}\right)^2}{2} = 108°$$

From this, it can be determined that the other two angles in the triangle are,

$$\frac{180 - 108}{2} = 36°$$

Repeating the same steps for the green triangle, half of the kite tile, letting the short side be a, the two long sides be b and c, and θ be the angle between the two long sides,

$$\cos\theta = \frac{(\emptyset x)^2 + (\emptyset x)^2 - x^2}{2 \cdot \emptyset x \cdot \emptyset x}$$

$$\cos\theta = \frac{2\emptyset^2 x^2 - x^2}{2\emptyset^2 x^2}$$

$$\cos\theta = 1 - \frac{1}{2\emptyset^2}$$

$$\cos\theta = 1 - \frac{1}{2\left(\frac{1+\sqrt{5}}{2}\right)^2}$$

$$\theta = \cos^{-1}\left(1 - \frac{4}{2\left(1+\sqrt{5}\right)^2}\right) = 36°$$

From this, it can be determined that the other two angles in the triangle are,

$$\frac{180 - 36}{2} = 72°$$

Using this information, all the angles in the tiles can be calculated, shown in Figure 4.

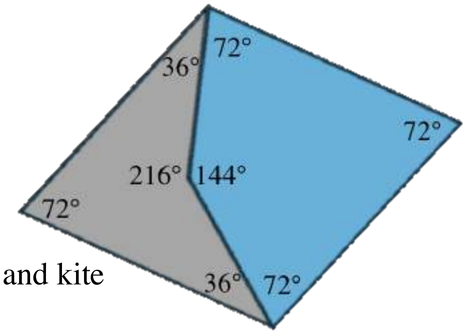


Figure 4: Angles of dart and kite tiles

There are matching rules that determine the “legal” placement of tiles to ensure the creation of a nonperiodic tiling. One method is to label the corners of the tiles with two different colours, as shown in Figure 5. The tilings must then be placed so that only corners with matching colours may be joined[6], for example, in the tiling shown in Figure 5. This means the tiles cannot be joined together to form a rhomb, and this makes sense since tesselations of rhombs would form a periodic tiling.

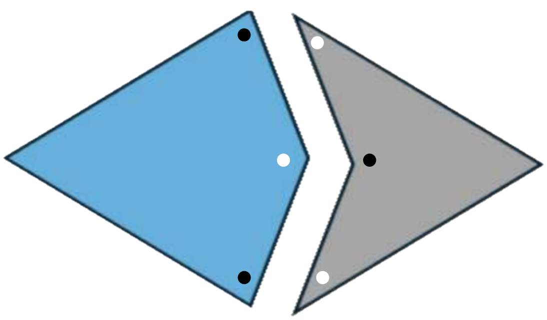


Figure 5.1: Kite and dart tiles with coloured corners

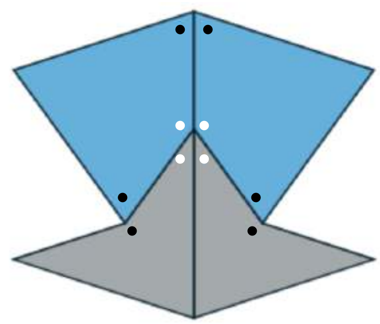


Figure 5.2: Tiling following matching rules

Another method to illustrate the matching rules, proposed by John Conway, an English mathematician, is to draw circular arcs of two different colours on each tile, as shown in Figure 6. To form a nonperiodic tiling, the tiles can only be placed so that arcs of the same colour join together.[7]

[6] Ibid., 7.
[7] Ibid.

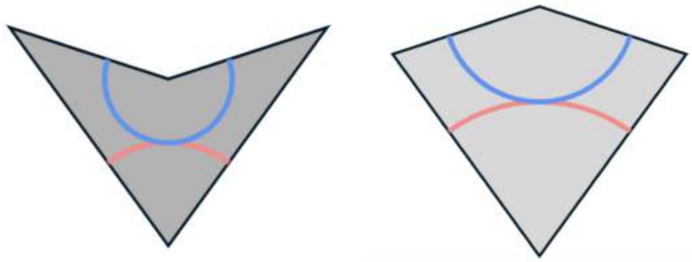


Figure 6: Dart and kite tiles with arcs displayed

Figure 7 shows an example of a tiling following the matching rules. It can be seen that the end of the arcs on the tiles are joined.

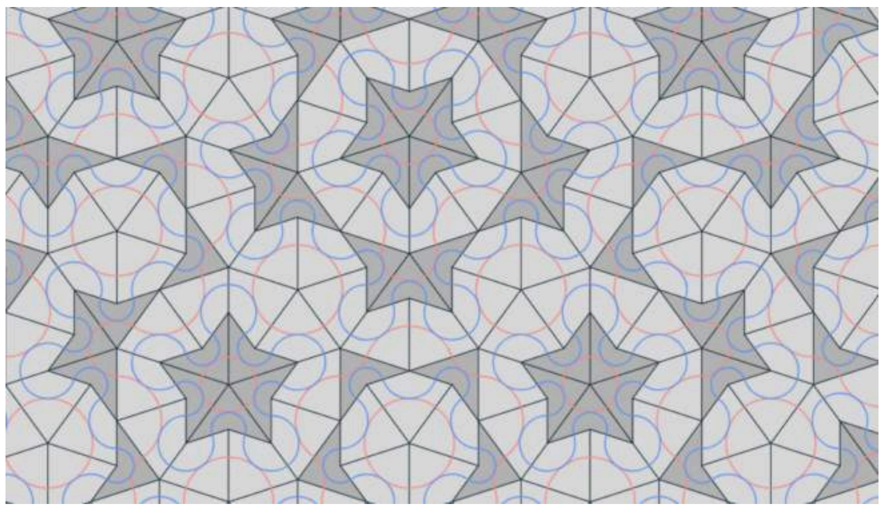


Figure 7: Penrose tiling following matching rules with arcs displayed

It should be noted, however, that these matching rules do not guarantee an infinite, legal tiling of the plane. A tile could be placed, following the matching rules, that leads to a situation many tiles later in which a tile can no longer be legally placed. As of now, no local rules, or rules determined by the local envrionment of the tiling, have been found that determine the placement of tiles to allow the legal tiling of an infinite plane.[8] For

[8] Laura Effinger-Dean. "The Empire Problem in Penrose Tilings" (Bachelor's Thesis, Williams College, 2006). 9-10.

example, Figure 8 shows several dart and kite tiles placed in accordance with the matching rules; it can be seen that arcs of the same colour join. Although additional tiles can continue to be placed on the perimeter of the current tiling, it can be seen that there is no way to arrange dart and kite tiles within the central gap surrounded by tiles (outlined in red), without creating overlapping regions. As tiles cannot overlap and there can be no gaps within an infinite tiling of a plane, the arrangement of tiles in Figure 8, despite following matching rules, makes a legal, infinite tiling impossible.

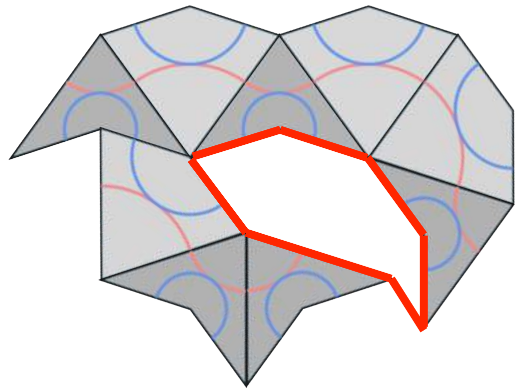


Figure 8: Penrose tiling in accordance with matching rules

Deflation and Inflation

Penrose discovered that every finite patch of Penrose tiles can infinitely be replaced with smaller or larger tiles, creating a new tiling; Conway coined this phenomenon as "deflation" and "inflation". A tiling is inflated, or replaced with larger tiles, by cutting all the tiles in half and joining the short edges of the original pieces together.[9] Deflation is the opposite process and new tilings can be created infinitely through substitution of larger tiles with smaller ones. When a tiling is inflated, the lengths of the sides of the tiles will increase by a factor of the golden ratio, $\emptyset$, and conversely, when a tiling is deflated, lengths will decrease by a factor of $\emptyset$. Figure 9 shows one generation of deflation of a dart and kite tile.

[9] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers* (Washington: Mathematical Association of America, 1997), 8.

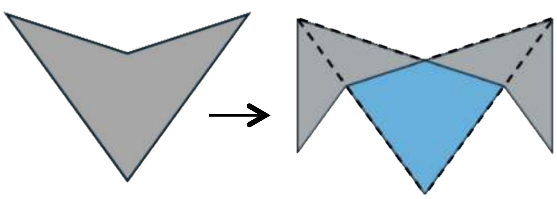


Figure 9.1: Deflation of a dart tile

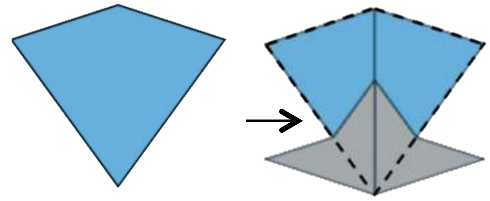


Figure 9.2: Deflation of a kite tile

Theorem 1: When any legal tiling is deflated, another legal tiling is produced.

Proof. There are only 7 ways tiles legally arrange around a vertex.[10] These 7 arrangements deflate so that the tiling around the original vertex is another one of the 7 legal arrangements, shown by the diagrams in Figure 10. This proves that any legal tiling arranged around a vertex can be deflated into a new legal tiling.

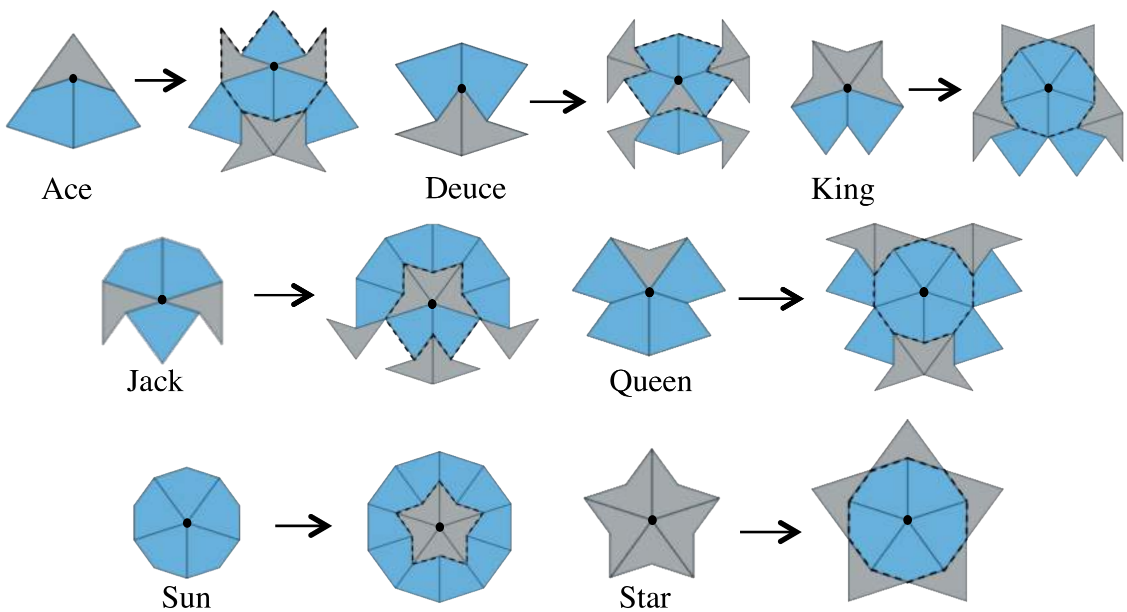


Figure 10: Arrangement of tiles around the seven types of vertices

This completes the proof.

[10] Laura Effinger-Dean. "The Empire Problem in Penrose Tilings" (Bachelor's Thesis, Williams College, 2006). 20.

Non-periodicity of Penrose Tilings

Penrose tilings are non-periodic, meaning that they lack translational symmetry. Penrose suggests that it can be proved that an infinite Penrose tiling, a Penrose tiling consisting of an infinite number of tiles, is non-periodic if the ratio of the two types of tiles is an irrational number.[11] To investigate this, I decided to use deflation, the process that replaces each tile with a number of smaller tiles; therefore, deflating a dart or kite tile an infinite number of times will result in an infinite Penrose tiling. Also, because a dart or kite tile is legal, the deflated infinite tiling will also be legal, as proved earlier. I deflated a dart and kite tile 4 generations each, shown in Figure 11, noting down the resulting number of dart and kite tiles after each deflation; results are shown in Table 1.

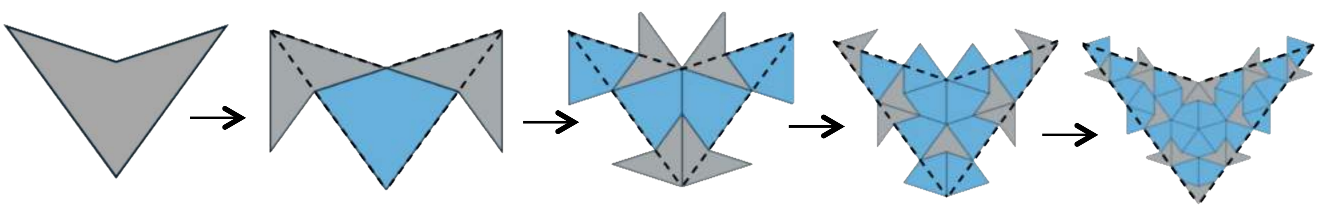


Figure 11.1: First 4 deflations of dart tile

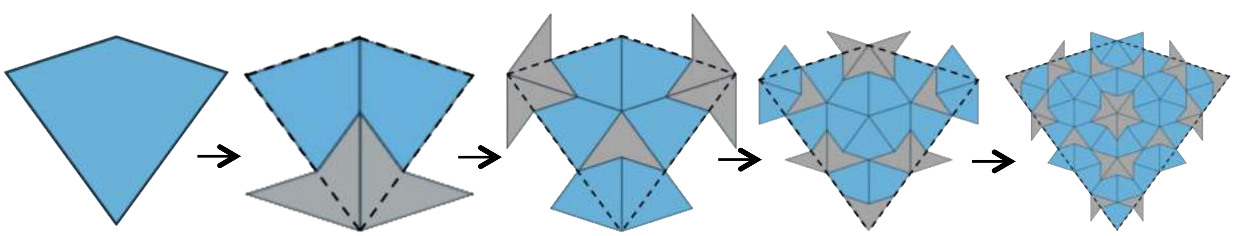


Figure 11.2: First 4 deflations of kite tile

[11] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers* (Washington: Mathematical Association of America, 1997), 7.

Number of Deflations	1		2		3		4	
Type of Tile	Dart	Kite	Dart	Kite	Dart	Kite	Dart	Kite
Number of tiles (When deflating dart)	1	1	2	3	5	8	13	21
Number of tiles when (When deflating kite)	1	2	3	5	8	13	21	34

Table 1: Recorded number of dart and kite tiles after each deflation

A familiar pattern can be observed from these data; for both the dart and kite tile, the sequence made up of the numbers of darts and kites after successive deflations seems to mimic the Fibonacci sequence, a series of numbers where each term is the sum of the two terms before it.[12] This led me to come up with the following theorem.

Theorem 2: The sequence created with alternate values of dart and kite tiles remaining after infinite generations of deflation mimics the Fibonacci sequence.

Proof.

I will prove the theorem through induction.

Although deflating a dart or kite tile both seem to result in the Fibonacci sequence, I will be using the sequence resulting from deflating a dart tile. Let this sequence be

[12] Pravin Chandra and Eric W. Weisstein. "Fibonacci Number." From Wolfram MathWorld. Accessed January 18, 2018. http://mathworld.wolfram.com/FibonacciNumber.html.

represented by t_n. The Fibonacci sequence can be represented by the formula $F_{n+1} = F_n + F_{n-1}$. Therefore, let the proposition be that $t_{n+1} = t_n + t_{n-1}$ is true for all $n \in \mathbb{Z} > 2$.

Base case ($n = 2$)*:*

By using values from the table,

$$\text{Left hand side: } t_{n+1} = t_{2+1} = t_3 = 2$$

$$\text{Right hand side: } t_n + t_{n-1} = t_2 + t_{2-1} = t_2 + t_1 = 1 + 1 = 2$$

Left hand side = Right hand side

$\therefore$ Proposition is true for $n = 2$

Inductive Hypothesis: Assume that the claim holds for $n = k$. Therefore,

$$t_{k+1} = t_k + t_{k-1} \ldots (1)$$

Inductive Step:

We want to prove that the proposition holds for $n = k + 1$. Substituting $n = k + 1$ into the proposition equation,

$$t_{k+1+1} = t_{k+1} + t_k$$

$$t_{k+2} = t_{k+1} + t_k \ldots (2)$$

Depending on whether the term of the sequence is odd or even, the type of tile represented will differ. As seen in Table 1, every odd numbered term represents the number of dart tiles, whilst even numbered terms represent the number of kite tiles. Deflating a dart tile will result in 1 dart and 1 kite tile, while deflating a kite tile results is 1 dart and 2 kites, as shown previously in Figure 10. Mathematically,

$$D'(deflated\ dart\ tile) = D + K \ \ldots (3)$$

$$K'(deflated\ kite\ tile) = D + 2K \ldots (4)$$

where D' = deflated dart tile, K'= deflated kite tile, D = dart tile and K = kite tile.

1st case: k is an even number. $n = k + 1$, therefore, is an odd number. When n is odd, t_n represents the number of dart tiles. From (3) and (4) we see that the deflation of every dart or kite tile results in one dart tile. Therefore, the number of dart tiles after one generation of deflation will simply be the original number of dart and kite tiles added together. Therefore, when k is an even number, t_k is the sum of the previous two terms, or $t_k = t_{k-2} + t_{k-1}$. As $k + 2$ is also an even number, it holds that $t_{k+2} = t_k + t_{k+1}$, which equals (2), and therefore, the proposition is true for $n = k + 1$ when k is an even number.

2nd case: k is an odd number. $n = k + 1$, therefore, is an even number. When n is even, t_n represents the number of kite tiles. From (3) and (4) we see that the deflation of every dart tile results in one dart tile, and every deflation of a kite tile results in two kite tiles. Therefore, when k is an odd number, t_k is the sum of the original number of dart tiles, which is represented by t_{k-3}, and double the original number of kite tiles, represented by t_{k-2}. Therefore, $t_k = t_{k-3} + 2t_{k-2}$. When k is an odd number, $k + 2$ is also an odd number, and therefore, substituting $k + 2$ into k, we get

$$t_{k+2} = t_{k+2-3} + 2t_{k+2-2}$$

$$t_{k+2} = t_{k-1} + 2t_k \ldots(5)$$

Substituting (1) into (2),

$$t_{k+2} = t_k + t_{k-1} + t_k$$

$$t_{k+2} = t_{k-1} + 2t_k$$

which equals (5), therefore, the proposition is true for $n = k + 1$ when k is an odd number.

Therefore, the proposition, $t_{n+1} = t_n + t_{n-1}$, is true for all $n \in \mathbb{Z} > 2$.

This completes the proof.

The successive terms in this sequence, which mimics the Fibonacci sequence, represents the ratio of kite to dart tiles. For example, the first two terms show that after one deflation of the dart tile, the ratio of kites to darts is 1:1. Penrose suggested that as the number of deflations approach infinity, the ratio will approach the golden ratio, which leads me to the following theorem.

Theorem 3: The ratio of successive terms in the Fibonacci sequence will approach the golden ratio, $\frac{1+\sqrt{5}}{2}$, as the number of terms approaches infinity.

Proof.

The Fibonacci sequence is represented by the equation,

$$F_{n+1} = F_n + F_{n-1} \dots (1)$$

Successive terms in the sequence can be represented by F_n and F_{n+1}, and therefore, their ratio, by $\frac{F_{n+1}}{F_n}$. Therefore, as the number of terms approaches infinity, $n \to \infty$, the ratio is represented as,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

Substituting in (1) gives,

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} \frac{F_n + F_{n-1}}{F_n}$$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} 1 + \frac{F_{n-1}}{F_n}$$

$$\lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \lim_{n\to\infty} 1 + \lim_{n\to\infty} \frac{F_{n-1}}{F_n} \dots (2)$$

$$Let\ x = \lim_{n\to\infty} \frac{F_{n+1}}{F_n}$$

$$\lim_{n\to\infty} \frac{F_n}{F_{n+1}} = \lim_{n\to\infty} \frac{F_{n-1}}{F_n} = \frac{1}{x}$$

Substituting x into (2),

$$x = 1 + \frac{1}{x}$$

$$x^2 = x + 1$$

$$x^2 - x - 1 = 0$$

$$x = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore \lim_{n\to\infty} \frac{F_{n+1}}{F_n} = \frac{1+\sqrt{5}}{2}$$

End of proof.

Since the golden ratio is an irrational number (see Appendix B for proof), the ratio of kites to darts is irrational, proving the nonperiodicity of an infinite plane of Penrose tilings.

Five-fold Symmetry of Closed Loops

Symmetry is a defining characteristic of periodic tilings as the ordered patterns clearly display translational and rotational symmetry. For example, Figure 12[13] displays a periodic tiling. On top of translational symmetry, this tiling displays

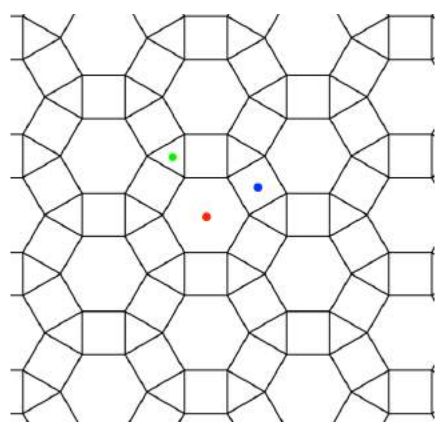


Figure 12: Periodic tiling displaying rotational symmetries

[13] Figure 12: Denis Zorin. The Archimedean tilings". 2003." Mass: Multiresolutional Adaptive Solid Subdivision. https://www.researchgate.net/figure/2564710_Figure-13-The-Archimedean-tilings-dual-to-Laves-Image-curtesy-of-Denis-Zorin

3-fold (around the green point), 4-fold (around the blue point) and 6-fold (around the red point), rotational symmetry.

It may be assumed that non-periodic tilings, such as Penrose tilings, would not display symmetry of any kind. Although it is true that nonperiodic tilings lack translational symmetry, they can display rotational symmetry. When dart and kite tiles are specifically arranged, Penrose tilings can display five-fold rotational symmetry, meaning that the tiling can be mapped onto itself when rotated an angle of $\frac{2\pi}{5} rad$ or $\frac{360°}{5} = 72°$. Two of the seven arrangements of tiles around a vertex exhibit this rotational symmetry, the sun and star, displayed in Figure 13 with arcs displayed.

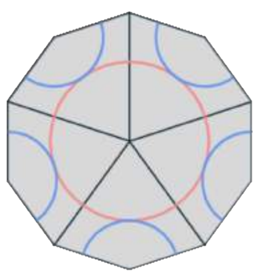


Figure 13.1: Sun

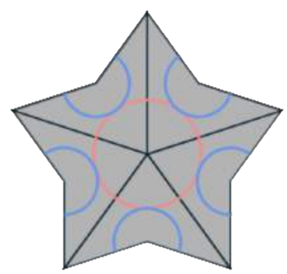


Figure 13.2: Star

Notice that both of these arrangements have a closed loop (pink). Conway proved that whenever the coloured arcs close in a loop in a Penrose tiling, the region within the loop will display five-fold rotational symmetry. He stated that if you inflate any closed loop a number of times, you will end up with the closed loop in the sun or star arrangement, and inflating either of these arrangements one more time will lead to the closed loop disappearing altogether.[14] To reverse this process, I deflated both the sun and star arrangement a number of times, as shown in Figure 14.

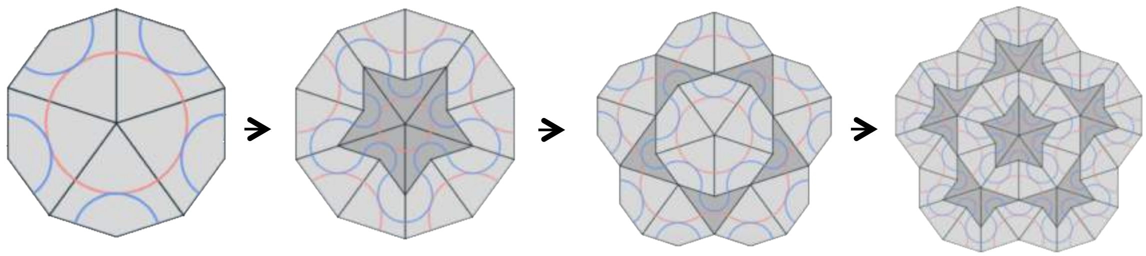


Figure 14: Deflation of sun arrangement

It can be seen that deflating the sun or star tile will create new tilings in which closed loops can be seen, therefore, displaying five-fold rotational symmetry. To preserve the pentagonal symmetry, tiles can only be arranged around the vertex in the patterns created through deflation of the sun and star arrangements. This pattern is known as the infinite sun and star patterns as they are determined to infinity[15].

Self Similarity of Penrose Tiles

A periodic tiling is self similar as any finite patch of the tiling will be repeated an infinite number of times in an infinite tiling. For example, Figure 15[16] displays a periodic tiling. The finite patch outlined in red will appear,

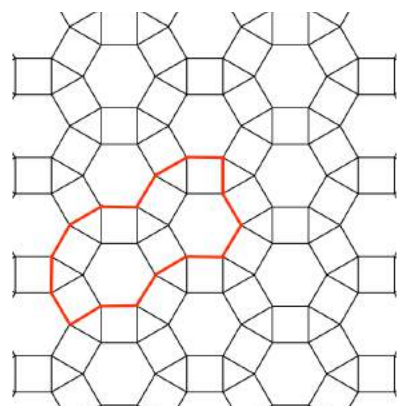


Figure 15: Self similar periodic tiling

[14] Branko Grünbaum and Geoffrey Colin Shephard. *Tilings and patterns*. (Mineola: Dover Publications, Inc., 2016).
[15] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers* (Washington: Mathematical Association of America, 1997), 11.
[16] Figure 15: Denis Zorin. The Archimedean tilings". 2003." Mass: Multiresolutional Adaptive Solid Subdivision. https://www.researchgate.net/figure/2564710_Figure-13-The-Archimedean-tilings-dual-to-Laves-Image-curtesy-of-Denis-Zorin

translated and reflected, an infinite number of times throughout the tiling if it were to be infinite.

The "local isomorphism theorem" by Penrose proves that this self similarity is seen in nonperiodic Penrose tiles, in that every finite region within a Penrose tiling is contained in, and appears an infinite number of times in every other tiling.[17] For example, Figure 16 displays a patch of Penrose tilings. It can be seen that the region outlined by the red circle appears twice more (outlined by the dotted red line), within the patch of tiling. For an infinite tiling, the outlined region will appear an infinite number of times.

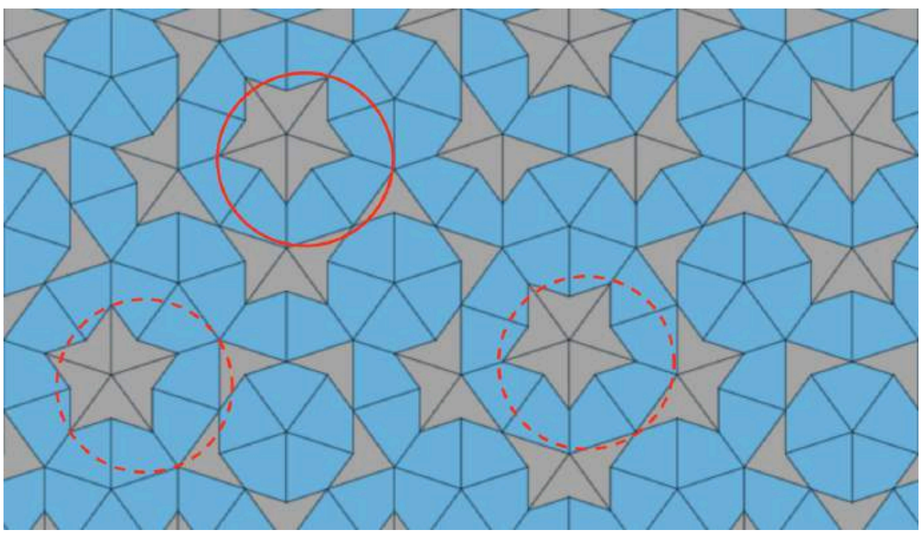


Figure 16: Local isomorphism displayed within a finite patch of a Penrose tiling

Conway states that these circular regions will always appear within a distance from each other; he proved that the distance from the perimeter of one region with a diameter of d will never be more than d times half of the cube of the golden ratio from the next identical region[18].

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[17] Ibid., 9-10.
[18] Ibid.

In an attempt to find a distance myself, within which two identical circular regions may be found, let there be a finite, circular region of diameter, dx where d is a constant and x is the length of the longer side of a dart or kite tile. Continuously inflating this region, will eventually result in the region containing a single vertex; an example is given in Figure 17. When the circular region is inflated once, the lengths of the sides of the tiles, and therefore, x, will increase by a factor of the golden ratio, $\emptyset$; therefore, defining x to be the length of the long side of the tile in the new inflated tiling will lead to the diameter being divided by a factor of $\emptyset$ to be represented as $\frac{d}{\emptyset}x$. If we inflate the tile again, the new diameter will be $\frac{d}{\emptyset^2}x$ where x is the value of the long side of a tile in the new inflated tiling. In general, for n deflations, the diameter will be $\frac{d}{\emptyset^n}x$ with the x value being the long side of a tile in the current tiling. If this value for the diameter is less than the short side of a tile in the current tiling, represented by $\frac{x}{\emptyset}$, or mathematically,

$$\frac{d}{\emptyset^n}x \leq \frac{x}{\emptyset}$$

the maximum number of vertices that can be within the finite region will be one.

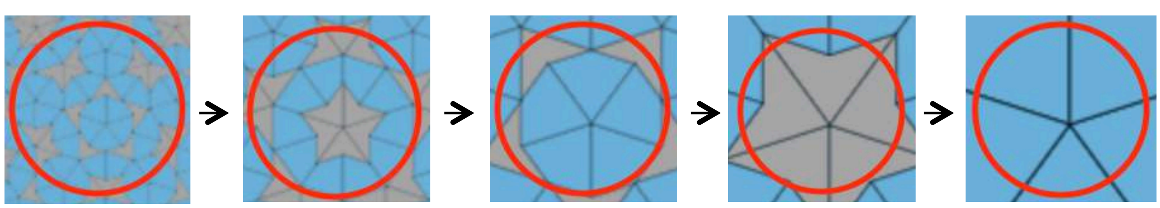


Figure 17: Inflation of a finite patch of Penrose tiling

As all finite patches of tiling within a region, will inflate to leave one vertex within that region, finding the minimum distance between two vertices will allow me to find the minimum distance between two identical finite patches of tiling. As discussed before, there are seven ways to arrange tiles around a vertex, and therefore, we must consider each of these arrangements when finding the distance between two vertices.

First let us consider the ace. Displayed in Figure 18.1 are the two closest possible ace arrangements, outlined in red. Figure 18.2 displays the necessary lengths and angles to calculate the minimum distance, solid red line, between the two vertices.

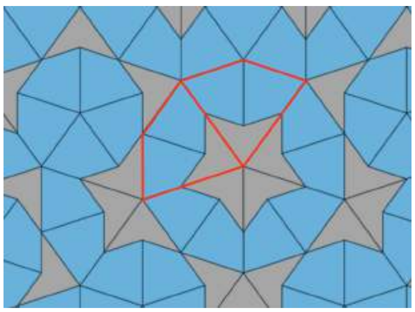


Figure 18.1: Closest two ace arrangements

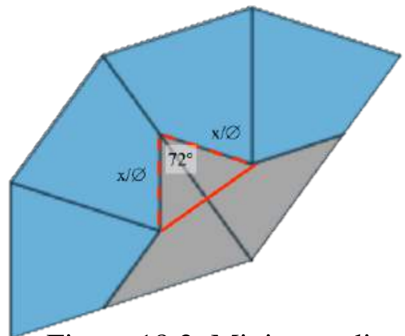


Figure 18.2: Minimum distance between ace vertices

Using the cosine rule,

$$a^2 = b^2 + c^2 - 2bc\cos\theta,$$

where a = minimum distance, $b = c$ = dotted red lines, and θ = angle between dotted red lines.

$$a^2 = \left(\frac{x}{\emptyset}\right)^2 + \left(\frac{x}{\emptyset}\right)^2 - 2\left(\frac{x}{\emptyset}\right)\left(\frac{x}{\emptyset}\right)\cos 72°$$

$$a^2 = \frac{2x^2}{\emptyset^2} - \frac{2x^2}{\emptyset^2}\cos 72°$$

$$a^2 = \frac{2x^2}{\emptyset^2}(1 - \cos 72°)$$

$$a = \sqrt{\frac{2x^2}{\emptyset^2}(1 - \cos 72°)}$$

$$a = \frac{x}{\emptyset}\sqrt{2 - 2\cos 72°} \approx 0.727x$$

I repeated this for the rest of the possible arrangements of tiles around a vertex: the deuce (Figure 19), the king (Figure 20), the jack (Figure 21), the queen (Figure 22), the sun (Figure 23) and the star (Figure 24).

<u>Deuce:</u>

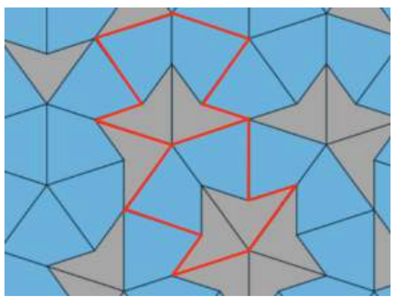


Figure 19.1: Closest two deuce arrangements

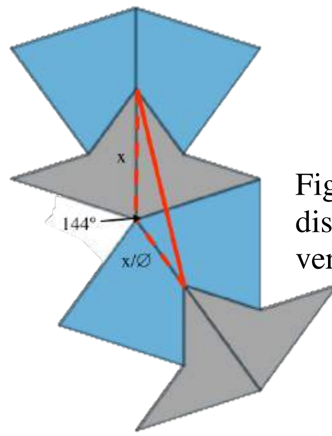


Figure 19.2: Minimum distance between deuce vertices

$$a^2 = (x)^2 + \left(\frac{x}{\emptyset}\right)^2 - 2(x)\left(\frac{x}{\emptyset}\right)\cos 144°$$

$$a^2 = \frac{\emptyset^2 x^2 + x^2}{\emptyset^2} - \frac{2x^2}{\emptyset}\cos 144°$$

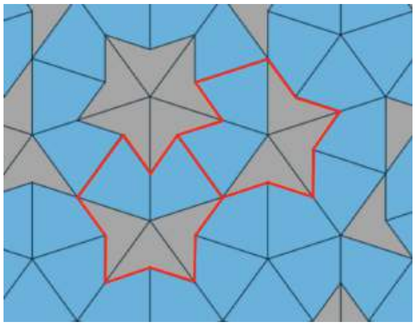
$$a^2 = \frac{(\emptyset^2 + 1)x^2}{\emptyset^2} - \frac{2x^2}{\emptyset}\cos 144°$$

$$a^2 = x^2\left(\frac{(\emptyset^2 + 1)}{\emptyset^2} - \frac{2}{\emptyset}\cos 144°\right)$$

$$a = \sqrt{x^2\left(\frac{(\emptyset^2+1)}{\emptyset^2} - \frac{2}{\emptyset}\cos 144°\right)}$$

$$a = x\sqrt{\left(\frac{(\emptyset^2 + 1)}{\emptyset^2} - \frac{2}{\emptyset}\cos 144°\right)} \approx 1.62x$$

King:



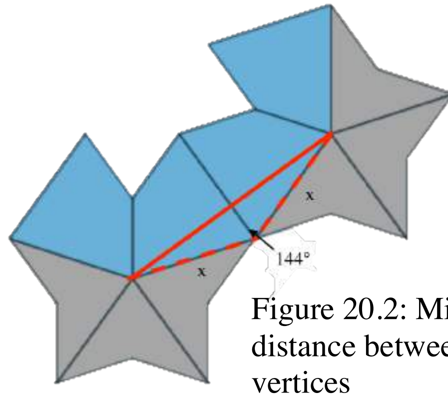


Figure 20.2: Minimum distance between king vertices

Figure 20.1: Closest two king arrangements

$$a^2 = (x)^2 + (x)^2 - 2(x)(x)\cos 144°$$

$$a^2 = 2x^2 - 2x^2 \cos 144°$$

$$a^2 = x^2(2 - 2\cos 144°)$$

$$a = \sqrt{x^2(2 - 2\cos 144°)}$$

$$a = x\sqrt{2 - 2\cos 144°} \approx 1.90x$$

Jack:

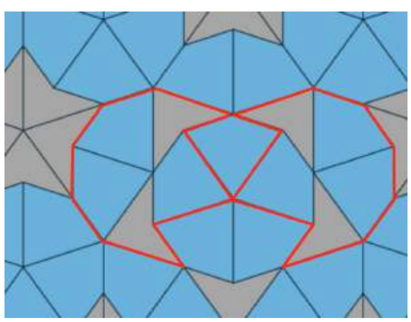


Figure 21.1: Closest two jack arrangements

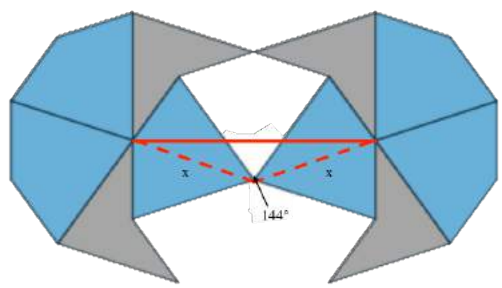


Figure 21.2: Minimum distance between jack vertices

Same calculations as for the king vertices, therefore,

$$a = x\sqrt{2 - 2\cos 144°} \approx 1.90x$$

Queen:

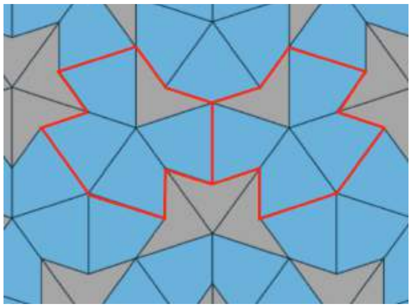


Figure 22.1: Closest two queen arrangements

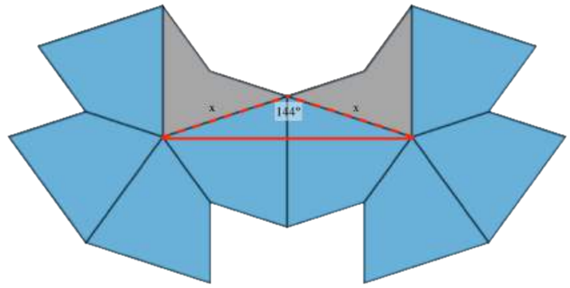


Figure 22.2: Minimum distance between queen vertices

Same calculations as for the king vertices, therefore,

$$a = x\sqrt{2 - 2\cos 144°} \approx 1.90x$$

Sun:

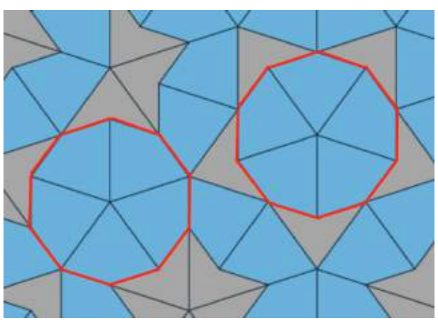


Figure 23.1: Closest two sun arrangements

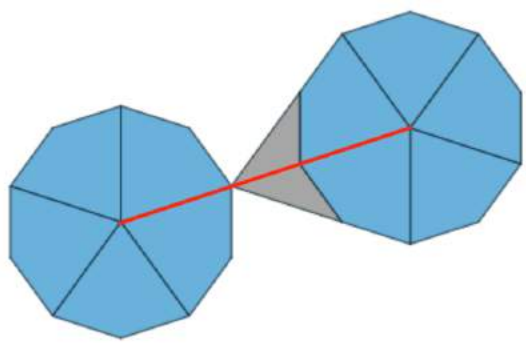


Figure 23.2: Minimum distance between sun vertices

The minimum distance between the sun vertices is made up of two long sides and a short side of a tile, therefore, the distance is,

$$2x + \frac{x}{\emptyset} \approx 2.62x$$

Star:

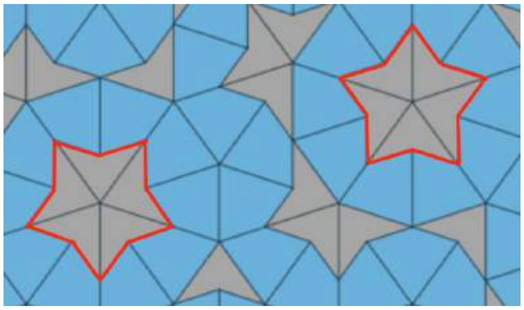


Figure 24.1: Closest two star arrangements

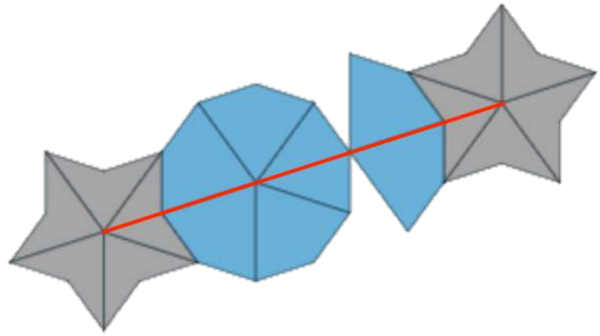
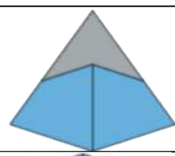
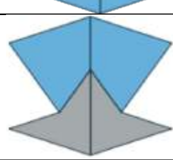
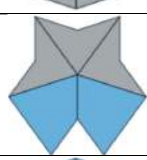
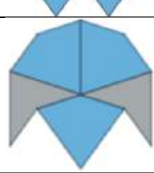
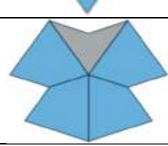


Figure 24.2: Minimum distance between star vertices

The minimum distance between the star vertices are made up of three long sides and two short side of a tile, therefore, the distance is,

$$3x + \frac{2x}{\emptyset} \approx 4.24x$$

I have summarized my findings in Table 2.

Vertex	Distance (in terms of x) between vertices to 3 s.f.
	$0.727x$
	$1.62x$
	$1.90x$
	$1.90x$
	$1.90x$

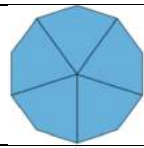
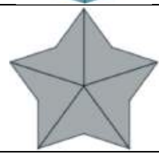
	$2.62x$
	$4.24x$

Table 2: Minimum distance between matching vertices

If we wanted to find the distance of the original patch of tilings from an identical region, we will have to deflate the vertices back to its orginal state. While inflating the tiles once caused the diameter to decrease by a factor of $\emptyset$, deflating will do the opposite and increase the diameter by a factor of $\emptyset$. Therefore, when deflating the region n times to revert it to its original state, the diameter will be increased by a factor of $\emptyset^n$, and so therefore, the distances found for the vertices will also need to be multiplied by $\emptyset^n$ where n is the number of inflations taken to be left with a single vertex within the finite region.

Conclusion: Penrose Tilings as a Model for Quasicrystals

Tilings often serve as geometric models for crystallography with each tile representing a "unit cell".[19] However, the basic definition of crystals had always been that they are ordered and periodic.[20] Although nonperiodic, Penrose tilings share a number of characteristics with periodic tilings, such as rotational symmetry and self similarity. Therefore, it may not be so surprising that atomic structures resembling nonperiodic Penrose tiling have emerged in alloys and minerals. The first of these structures, an

[19] Laura Effinger-Dean. "The Empire Problem in Penrose Tilings" (Bachelor's Thesis, Williams College, 2006). 8.
[20] Daniel Oberhaus. "Quasicrystals Are Nature's Impossible Matter". Vice Motherboard. 2015. https://motherboard.vice.com/en_us/article/4x3me3/quasicrystals-are-natures-impossible-matter.

aluminium manganese alloy, was discovered in 1984, and displayed fivefold symmetry. This discovery defied the definition of a crystal held at that time; only 2, 3, 4 and 6 fold rotational symmetries had ever been observed in solid materials. Since the first discovery, similar nonperiodic structures were being found in other alloys and the structures came to be known as quasicrystals, as they display quasiperiodicity or, irregular periodicity.[21] In 2009, a mineral, icosahedrite, was discovered suggesting the natural formation of quasicrystals.[22]

All Penrose tilings display fivefold rotational symmetry, leading them to become a model for understanding the structure and properties of quasicrystals. However, as mentioned before, Penrose tilings do not have local rules that determine how tiles should be placed to ensure an infinite tiling; this limits their ability to explain how nonperiodic quasicrystals form and grow. There are many questions still to be answered. Are there other geometric models that can explain these unorthodox crystals? As these structures grow, what rules determine how atoms are placed, ensuring nonperiodicity? What physical forces, fields are involved in their formation?

[21] Martin Gardner, *Penrose Tiles to Trapdoor Ciphers* (Washington: Mathematical Association of America, 1997), 25.
[22] Daniel Oberhaus. "Quasicrystals Are Nature's Impossible Matter". Vice Motherboard. 2015. https://motherboard.vice.com/en_us/article/4x3me3/quasicrystals-are-natures-impossible-matter.

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Figures 1, 12 & 15: Zorin, Denis. 2003."The Archimedean tilings". Mass: Multiresolutional Adaptive Solid Subdivision. https://www.researchgate.net/figure/2564710_Figure-13-The-Archimedean-tilings-dual-to-Laves-Image-curtesy-of-Denis-Zorin

Austin, David. "Penrose Tiles Talk Across Miles." American Mathematical Society. Accessed January 08, 2018. http://www.ams.org/samplings/feature-column/fcarc-penrose.

Figure 2: "A P1 tiling using Penrose's original set of six prototiles". 2009. Wikipedia. https://en.wikipedia.org/wiki/Penrose_tiling

Effinger-Dean, Laura. "The Empire Problem in Penrose Tilings". Bachelor's Thesis, Williams College, 2006.

Chandra, Pravin, and Weisstein, Eric W.. "Fibonacci Number." From Wolfram MathWorld. Accessed January 18, 2018. http://mathworld.wolfram.com/FibonacciNumber.html.

Grünbaum, Branko and Shephard, Geoffery Colin. *Tilings and patterns*. Mineola: Dover Publications, Inc., 2016.

Oberhaus, Daniel. "Quasicrystals Are Nature's Impossible Matter." Vice Motherboard. May 03, 2015. Accessed January 08, 2018. https://motherboard.vice.com/en_us/article/4x3me3/quasicrystals-are-natures-impossible-matter.

Appendices

Appendix A

Below is the code I wrote using JavaScript in order to create and manipulate diagrams of Penrose tilings.

```
<!DOCTYPE HTML>
<html>
  <head>
    <style>
      body {
        margin: 0px;
        padding: 0px;
      }
    </style>
  </head>
  <body>
    <form>
        <p>

            <input type="button" name="deflate1" value="Deflate"
onclick="deflate()">
            <input type="button" name="rotate1" value="Rotate Right"
onclick="rotateRight()">
            <input type="button" name="rotate2" value="Rotate Left"
onclick="rotateLeft()">
            <input type="button" name="zoom" value="Zoom In" onclick="zoomIn()">
            <input type="button" name="zoom2" value="Zoom Out" onclick="zoomOut()">
            <input type="button" name="arc" value="Draw Arcs" onclick="drawArc()">
            <input type="button" name="erasearc" value="Erase Arcs"
onclick="eraseArc()">
            <input type="button" name="delete" value="Delete Shape"
onclick="changeToDeleteMode()">
        </p>

    </form>
    <canvas id="canvas" width="1000" height="600"></canvas>
    <script>

        var canvas = document.getElementById('canvas');
        var context = canvas.getContext('2d');


        var info=[[],[],[],[]];
        var newInfo=[[],[],[],[]];
        var totalTiles=1
        var length=200
        var phi=(1+Math.sqrt(5))/2
        var deleteMode=0

        var colorKite="#6AB1DA"
        var colorDart="#a6a6a6"
        info[0][0]=500
        info[1][0]=400
        info[2][0]=270
        info[3][0]=1
        drawTiles()


        function drawTiles() {
            context.clearRect(0,0,1000,1000)
            for (var i=0; i<totalTiles; i++) {
```

```
                var x1=info[0][i]
                var y1=info[1][i]
                var x2=x1+length*Math.cos((info[2][i]-36)*Math.PI/180)
                var y2=y1+length*Math.sin((info[2][i]-36)*Math.PI/180)
                var x3=x1+length*Math.cos(info[2][i]*Math.PI/180)
                var y3=y1+length*Math.sin(info[2][i]*Math.PI/180)
                if (info[3][i]==1) {
                    x3=x1+length/phi*Math.cos(info[2][i]*Math.PI/180)
                    y3=y1+length/phi*Math.sin(info[2][i]*Math.PI/180)

                }
                var x4=x1+length*Math.cos((info[2][i]+36)*Math.PI/180)
                var y4=y1+length*Math.sin((info[2][i]+36)*Math.PI/180)


                context.beginPath();
                context.moveTo(x1,y1);
                context.lineTo(x2,y2);
                context.lineTo(x3,y3);
                context.lineTo(x4,y4);
                context.closePath();
                context.fillStyle=colorKite;
                if (info[3][i]==1) {context.fillStyle=colorDart}
                context.fill()
                context.lineWidth=length/90
                context.strokeStyle="#041C2A";
                context.stroke();
                context.write

            }
        }

        function deflate(){
            var tempData=[[],[],[],[]]
            for (var i=0; i<totalTiles; i++) {
                tempData[0][i]=info[0][i]
                tempData[1][i]=info[1][i]
                tempData[2][i]=info[2][i]
                tempData[3][i]=info[3][i]
            }
            var t=0; //total number of new shapes
            for (var i=0; i<totalTiles; i++) {
                if (tempData[3][i]==1) {
                    info[0][t]=tempData[0][i]
                    info[1][t]=tempData[1][i]
                    info[2][t]=tempData[2][i]
                    info[3][t]=0
                    t++
                    info[0][t]=tempData[0][i]+length*Math.cos((tempData[2][i]-
36)*Math.PI/180)
                    info[1][t]=tempData[1][i]+length*Math.sin((tempData[2][i]-
36)*Math.PI/180)
                    info[2][t]=tempData[2][i]+144
                    if (info[2][t]<0) {info[2][t]+=360}
                    if (info[2][t]>=360) {info[2][t]-=360}
                    info[3][t]=1
                    t++


info[0][t]=tempData[0][i]+length*Math.cos((tempData[2][i]+36)*Math.PI/180)

info[1][t]=tempData[1][i]+length*Math.sin((tempData[2][i]+36)*Math.PI/180)
                    info[2][t]=tempData[2][i]+216
                    if (info[2][t]<0) {info[2][t]+=360}
                    if (info[2][t]>=360) {info[2][t]-=360}
                    info[3][t]=1
                    t++
```

```
                }

                if (tempData[3][i]==0) {
                    info[0][t]=tempData[0][i]
                    info[1][t]=tempData[1][i]
                    info[2][t]=tempData[2][i]+36
                    if (info[2][t]<0) {info[2][t]+=360}
                    if (info[2][t]>=360) {info[2][t]-=360}
                    info[3][t]=1
                    t++

                    info[0][t]=tempData[0][i]
                    info[1][t]=tempData[1][i]
                    info[2][t]=tempData[2][i]-36
                    if (info[2][t]<0) {info[2][t]+=360}
                    if (info[2][t]>=360) {info[2][t]-=360}
                    info[3][t]=1
                    t++

                    info[0][t]=tempData[0][i]+length*Math.cos((tempData[2][i]-
36)*Math.PI/180)
                    info[1][t]=tempData[1][i]+length*Math.sin((tempData[2][i]-
36)*Math.PI/180)
                    info[2][t]=tempData[2][i]+108
                    if (info[2][t]<0) {info[2][t]+=360}
                    if (info[2][t]>=360) {info[2][t]-=360}
                    info[3][t]=0
                    t++


info[0][t]=tempData[0][i]+length*Math.cos((tempData[2][i]+36)*Math.PI/180)

info[1][t]=tempData[1][i]+length*Math.sin((tempData[2][i]+36)*Math.PI/180)
                    info[2][t]=tempData[2][i]+252
                    if (info[2][t]<0) {info[2][t]+=360}
                    if (info[2][t]>=360) {info[2][t]-=360}
                    info[3][t]=0
                    t++
                }
            }

            totalTiles=t
            length=length/phi
            for (var i=0; i<totalTiles; i++) {
                for (var j=i+1; j<totalTiles; j++){

                    if (Math.abs(info[0][i]-info[0][j])<0.0000001 &&
Math.abs(info[1][i]-info[1][j])<0.0000001 && info[2][i]==info[2][j] &&
info[3][i]==info[3][j]){
                        info[0].splice(j,1)
                        info[1].splice(j,1)
                        info[2].splice(j,1)
                        info[3].splice(j,1)
                        totalTiles-=1;
                    }
                }
            }

            /*var numberKite=0
            var numberDart=0
            for (var i=0; i<totalTiles; i++) {
                if (info[3][i]==0){numberKite+=1}
                if (info[3][i]==1){numberDart+=1}
            }
            alert(numberKite)
            alert(numberDart)*/
            drawTiles()
```

```
        }

        function zoomIn() {
            for (var i=0; i<totalTiles; i++) {
                info[0][i]=500+(-500+info[0][i])*phi;
                info[1][i]=300+(-300+info[1][i])*phi;
            }
            length=length*phi;
            context.clearRect(0,0,1000,1000)
            drawTiles()
        }

        function zoomOut() {
            for (var i=0; i<totalTiles; i++) {
                info[0][i]=500+(-500+info[0][i])/phi;
                info[1][i]=300+(-300+info[1][i])/phi;
            }
            length=length/phi;
            context.clearRect(0,0,1000,1000)
            drawTiles()
        }

        function rotateRight(){
            for (var i=0; i<totalTiles; i++) {
                newX=info[0][i]-500
                newY=info[1][i]-300
                info[0][i]=500+((newX)*Math.cos(36*Math.PI/180)-
(newY)*Math.sin(36*Math.PI/180));

info[1][i]=300+((newY)*Math.cos(36*Math.PI/180)+(newX)*Math.sin(36*Math.PI/180));
                info[2][i]=info[2][i]+36;
                if (info[2][i]<0) {info[2][i]+=360}
                if (info[2][i]>=360) {info[2][i]-=360}
            }
            context.clearRect(0,0,1000,1000)
            drawTiles()
        }

        function rotateLeft(){
            for (var i=0; i<totalTiles; i++) {
                newX=info[0][i]-500
                newY=info[1][i]-300
                info[0][i]=500+((newX)*Math.cos(-36*Math.PI/180)-(newY)*Math.sin(-
36*Math.PI/180));
                info[1][i]=300+((newY)*Math.cos(-36*Math.PI/180)+(newX)*Math.sin(-
36*Math.PI/180));
                info[2][i]=info[2][i]-36;
                if (info[2][i]<0) {info[2][i]+=360}
                if (info[2][i]>=360) {info[2][i]-=360}
            }
            context.clearRect(0,0,1000,1000)
            drawTiles()
        }

        function drawRedArc(){
            for (var i=0; i<totalTiles; i++) {
                context.beginPath();
                context.strokeStyle="#f28d8d"
                context.lineWidth=length/50
                if (info[3][i]==0){
                    context.arc(info[0][i],info[1][i],length/phi,(info[2][i]-
36)*Math.PI/180,(info[2][i]+36)*Math.PI/180)
                }
                if (info[3][i]==1){
                    context.arc(info[0][i],info[1][i],(length/phi)/phi,(info[2][i]-
36)*Math.PI/180,(info[2][i]+36)*Math.PI/180)
                }
```

```
            context.stroke()
            }
        }

        function drawGreenArc(){
            for (var i=0; i<totalTiles; i++) {
                context.beginPath();
                context.strokeStyle="#6495ED"
                context.lineWidth=length/50
                if (info[3][i]==0){
                    x2=info[0][i]+length*Math.cos(info[2][i]*Math.PI/180)
                    y2=info[1][i]+length*Math.sin(info[2][i]*Math.PI/180)

context.arc(x2,y2,(length/phi)/phi,(info[2][i]+108)*Math.PI/180,(info[2][i]-
108)*Math.PI/180)
                }
                if (info[3][i]==1){
                    x2=info[0][i]+(length/phi)*Math.cos(info[2][i]*Math.PI/180)
                    y2=info[1][i]+(length/phi)*Math.sin(info[2][i]*Math.PI/180)

context.arc(x2,y2,((length/phi)/phi)/phi,(info[2][i]+72)*Math.PI/180,(info[2][i]-
72)*Math.PI/180)
                }
            context.stroke()
            }
        }

        function drawArc(){
            colorKite="#D8D8D8"
            colorDart="#b3b3b3"
            context.clearRect(0,0,1000,1000)
            drawTiles()
            drawRedArc()
            drawGreenArc()
        }

        function eraseArc(){
            colorKite="#6AB1DA"
            colorDart="#A4A9AC"
            context.clearRect(0,0,1000,1000)
            drawTiles()
        }


        canvas.addEventListener("mousedown", mouseDown, false);


        function distance(x1,y1,x2,y2){
            return Math.sqrt(Math.pow(x1-x2,2)+Math.pow(y1-y2,2))
        }

        var mouseX=0;
        var mouseY=0;
        function mouseDown(e){
            mouseX=e.pageX-canvas.offsetLeft
            mouseY=e.pageY-canvas.offsetTop
            deleteShape()
        }

        function deleteShape(){
            if (deleteMode==1) {
                var closestDistance=1000000
                var closestIndex=-1
                for (var i=0; i<totalTiles; i++) {

                    x=info[0][i]+(length/phi)*Math.cos(info[2][i]*Math.PI/180)
                    y=info[1][i]+(length/phi)*Math.sin(info[2][i]*Math.PI/180)
```

```
                    d=distance(mouseX,mouseY,x,y)
                    if (d<closestDistance){
                        closestDistance=d
                        closestIndex=i
                    }
                }
                if (closestIndex==-1){return;}
                else {
                    info[0].splice(closestIndex,1)
                    info[1].splice(closestIndex,1)
                    info[2].splice(closestIndex,1)
                    info[3].splice(closestIndex,1)
                }
                totalTiles-=1;
                context.clearRect(0,0,1000,1000)
                drawTiles()
            }
        }


        function changeToDeleteMode(){
            if (deleteMode==0) {deleteMode=1}
            else {deleteMode=0}
        }

    </script>
  </body>
</html>
```

Appendix B

Theorem: The golden ratio, $\frac{1+\sqrt{5}}{2}$, is an irrational number.

Proof.

First, I will prove that $\sqrt{5}$ is an irrational number through proof by contradiction.

Let us assume that $\sqrt{5} = \frac{p}{q}$ where $p, q \ \in \mathbb{Z}$ and $GCD(p, q) = 1$ (the greatest common divisor of p and q is 1).

$$p = \sqrt{5}q$$

$$p^2 = 5q^2$$

Therefore, $5|p^2$ (5 is a factor of p^2)

To determine whether 5 is a factor of p,

Let $p = 5a + b = \sqrt{5}q$ where $a, \square \in \mathbb{Z}$ and $b \neq 0$ ($b = 1, 2, 3, 4$)

Squaring both sides of $5a + b = \sqrt{5}q$ gives,

$$25a^2 + 10ab + b^2 = 5q^2$$

$$b^2 = 5q^2 - 25a^2 - 10ab$$

$$b^2 = 5(q^2 - 5a^2 - 2ab)$$

$\therefore 5|b^2$, however, $5 \nmid 1^2, 5 \nmid 2^2, 5 \nmid 3^2, 5 \nmid 4^2$, which is a contradiction.

$\therefore b$ must equal 0, and so $p = 5a$

$\therefore 5|p$

$\therefore 25|p^2$ and $p^2 = 5q^2$, so $25|5q^2$

$\therefore 5|q^2$, which means $5|q$

$\therefore GDC(p,q) \geq 5$, however, this contradicts $GDC(p,q) = 1$

$$\therefore \sqrt{5} \neq \frac{p}{q}$$

Therefore, $\sqrt{5}$ is an irrational number.

The sum of a rational number and irrational number is always irrational, as is the quotient of an irrational number and a rational number. Therefore, $\frac{1+\sqrt{5}}{2}$, the golden ratio, is an irrational number.

End of proof.